Extremal decomposition of the complex plane

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The talk is devoted to a few well-known an extremal problems in geometric function theory of a complex variable associated with estimates of the functionals defined on the systems of mutually non-overlapping domains [1–9]. An improved method is proposed for solving problems on extremal decomposition of the complex plane. And effective upper estimates for maximum of the products of inner radii of mutually non-overlapping domains for any fixed systems of points of the complex plane at all possible values of some parameter γ are obtained [6–9]. Also we established conditions under which the structure of points and domains is not important. In particular, we obtained full solution of an open problem about extremal decomposition of the complex plane with two free poles located on the unit circle [8].

Problem. [1] Consider the product

$$r^{\gamma}\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where B_0,..., B_n, $n \geqslant 2$, are pairwise non-overlapping domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1, n}$ and $\gamma \in (0, n]$ ($r(B, a)$ be an inner radius of the domain $B \subset \overline{\mathbb{C}}$ relative to a point $a \in B$). For all values of the parameter $\gamma \in (0, n]$ to show that it attains its maximum at a configuration of domains B_k and points a_k possessing rotational n-symmetry.

The proof is due to Dubinin for $\gamma = 1$ [1] and to Kuz'mina [2] for $0 < \gamma < 1$. Subsequently, Kovalev [3] solved this problem under the additional assumption that the angles between neighbouring line segments $[0, a_k]$ do not exceed $2\pi/\sqrt{\gamma}$.

Let

$$I_n^0(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{\gamma}{n^2}\right)^{n + \frac{\gamma}{n}}} \left(\frac{1 - \frac{\sqrt{\gamma}}{n}}{1 + \frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

Theorem 1. [8] Let $\gamma \in (1, 2]$. Then, for any different points a_1 and a_2 of the unit circle and any mutually non-overlapping domains B_0, B_1, B_2, $a_1 \in B_1 \subset \overline{\mathbb{C}}$, $a_2 \in B_2 \subset \overline{\mathbb{C}}$, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, the inequality

$$r^{\gamma}\left(B_0, 0\right) r\left(B_1, a_1\right) r\left(B_2, a_2\right) \leqslant I_2^0(\gamma) \left(\frac{1}{2}\left|a_1 - a_2\right|\right)^{2-\gamma}$$

is true. The sign of equality in this inequality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(4-\gamma)w^2+\gamma}{w^2(w^2-1)^2}dw^2.$$

Theorem 2. [8] Let $n \in \mathbb{N}$, $n \geqslant 3$, $\gamma \in (1, n]$. Then, for any system of different points $\{a_k\}_{k=1}^n$ of the unit circle and for any collection of mutually non-overlapping domains B_0, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1, n}$, the following inequality holds

$$r^{\gamma}(B_0, 0)\prod_{k=1}^{n} r(B_k, a_k) \leqslant \left(\sin\frac{\pi}{n}\right)^{n-\gamma}\left(I_2^0\left(\frac{2\gamma}{n}\right)\right)^{\frac{n}{2}}.$$

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