On the symmetry groups of the neighborly polytopes

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An $n$-polytope $P$ is said to be $k$-neighborly if any subset of $k$ or less vertices is the vertex set of a face of $P$. The polytopes that are $\left\lceil \frac{n}{2} \right\rceil$-neighborly are of particular interests and are called neighborly polytopes. They are very important objects in combinatorics because they are solutions of various extremal properties such as the upper bound predicted by Motzkin for maximal number of $i$-faces of an $n$-polytope with $m$ vertices. A classical example of a neighborly $n$-polytope with $m$ vertices is the cyclic polytope $C^n(m)$. The cyclic polytope $C^n(m)$ is the convex hull

\[ C^n(m) := \text{conv}\{\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_m)\}, \]

for $m$ distinct points $\gamma(t_i)$ with $t_1 < t_2 < \cdots < t_m$ on the moment curve which is a curve in $\mathbb{R}^n$ defined by $\gamma : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto \gamma(t) = (t, t^2, \ldots, t^n) \in \mathbb{R}^n$. The combinatorial class of $C^n(m)$ does not depend on the specific choices of the parameters $t_i$ due to Gale’s evenness condition.

If the number of vertices $m$ of a neighborly $n$-polytope is not greater than $n + 3$ then combinatorially the polytope is isomorphic to a cyclic polytope. However, there are many neighborly polytopes which are not cyclic. Barnette in 1981 constructed an infinite family of duals of neighborly $n$-polytopes by using an operation called ‘facet splitting’ and Shemer in 1982 introduced a sewing construction that allows to add a vertex to a neighborly polytope in such a way as to obtain a new neighborly polytope. Both constructions show that for a fixed $n$ the number of combinatorially different neighborly polytopes grows superexponentially with the number of vertices $m$, but our knowledge about the combinatorics of this important objects is considerable small. The number of combinatorial types of neighborly polytopes in dimensions 4, 5, 6 and 7 with ‘small’ number of vertices is extensively studied in the last decades and following these results we determine their symmetry groups which are found to be very diverse. There exist not only the examples of the neighborly polytopes with trivial symmetry groups or $\mathbb{Z}/2\mathbb{Z}$, but also those with relatively big number of symmetries which rises questions of constructions of the neighborly polytopes with nontrivial symmetry group in arbitrary dimension.