Having fun with designs

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The kind of design we explore is a finite 2-design: a point-block incidence structure where each block is a k-subset of points, and each pair of points lies in a constant number λ of blocks. We ask that the design admits a block-transitive group G of automorphisms which preserves also a nontrivial partition of the point set. One famous study of these designs, by Delandtsheer and Doyen in 1989 introduced two parameters, now called Delandtsheer–Doyen parameters, that linked the design structure with the point-partition. Another even earlier study, by Davies in 1987, showed that, if G is transitive on flags (incident point-block pairs) then the number of points is bounded above by some function of λ (but no function was specified).

Recently, with Alice Devillers, we have been exploring these two results.

For flag-transitive designs: we showed that $4\lambda^6$ could be taken for the Davies function – though this is not a tight upper bound. Moreover, while investigating possible examples with small parameters, we found a rather beautiful flag-transitive design where the point set is a 6×6 grid, the block size is 8, $\lambda = 4$, and the full automorphism group is the symmetric group S_6.

Then, with Carmen Amarra and Alice Devillers, while exploring bounds and several extreme cases of the Delandtsheer–Doyen parameters, we constructed a (probably infinite) new family of designs. Whether the family is infinite depends on the Bunyakovsky Conjecture about the prime numbers, and the number of examples in a certain range seems to be related to another number theoretic conjecture, the Bateman—Horn Conjecture. We are grateful to Sasha Zvonkin and Gareth Jones for their interest in the number theoretic puzzles our construction spawned: their computer enumerations related to these conjectures showed that there are $12,357,532$ designs in the family where the point-partition has classes of prime cardinality less than 10^8.