Dual bases and orthogonal polynomials

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Let $b_0, b_1, \ldots, b_n$ be linearly independent functions. Let us consider the linear space $\mathcal{B}_n$ generated by these functions with an inner product $\langle \cdot, \cdot \rangle : \mathcal{B}_n \times \mathcal{B}_n \to \mathbb{C}$. We say that the functions $D_n := \{d_0^{(n)}, d_1^{(n)}, \ldots, d_n^{(n)}\}$ form *a dual basis* of the space $\mathcal{B}_n$ with respect to the inner product $\langle \cdot, \cdot \rangle$, if the following conditions hold:

$$\begin{cases} \operatorname{span}\left\{d_0^{(n)}, d_1^{(n)}, \ldots, d_n^{(n)}\right\} = \mathcal{B}_n, \\ \left\langle b_i, d_j^{(n)} \right\rangle = \delta_{ij} \qquad (0 \le i, j \le n), \end{cases}$$

where $\delta_{ii} = 1$, and $\delta_{ij} = 0$ for $i \ne j$.

In general, the dual basis D_n can be found with $O(n^2)$ computational complexity. However, if the dual basis D_n is known, it is possible to construct the dual basis D_{n+1} faster, i.e., with $O(n)$ computational complexity. Dual bases have many applications in numerical analysis, approximation theory or in computer aided geometric design. For example, skillful use of these bases often results in less costly algorithms which solve some computational problems.

It is also important that dual bases are very closely related to orthogonal bases. In the first part of the talk, we present general results on dual bases. Next, we focus on some important families of polynomial dual bases and their connections with classical, discrete and q-orthogonal polynomials. For example, the so-called dual Bernstein polynomials are related to orthogonal Hahn, dual Hahn and Jacobi polynomials. Using some of these connections, one can find differential-recurrence formulas, differential equation or recurrence relation satisfied by dual Bernstein polynomials. There also exists a first-order non-homogeneous recurrence relation linking dual Bernstein and orthogonal Jacobi polynomials. When used properly, it allows to propose fast and numerically efficient algorithms for evaluating all $n+1$ dual Bernstein polynomials of degree n with $O(n)$ computational complexity.