# Hardy–Littlewood–Sobolev inequality for $p = 1$

Dmitriy Stolyarov
*St. Petersburg State University*
d.m.stolyarov@spbu.ru

The Hardy–Littlewood–Sobolev inequality $\|I_\alpha f\|_{L_q(\mathbb{R}^d)} \lesssim \|f\|_{L_p(\mathbb{R}^d)}$, where $I_\alpha$ is the Riesz potential of order $\alpha$ and $1/p - 1/q = \alpha/d$, fails at the endpoint $p = 1$. I will show two ways to make the inequality true in this case. One way is to impose further restrictions on $f$ (like $f$ is a divergence free vector field), this way is related to the so-called Bourgain–Brezis inequalities. Another way is to replace the $L_q$-norm on the left hand side by an expression that has the same homogeneity as the $L_q$-norm, but possesses additional cancellations. The phenomenon has an analog in the world of martingale transforms, whose consideration suggests the right way to install induction on scales that proves those HLS inequalities for $p = 1$.