Balanced Equi-$n$-squares

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We define a $d$-balanced equi-$n$-square $L = (l_{ij})$, for some divisor $d$ of $n$, as an $n \times n$ matrix containing symbols from $\mathbb{Z}_n$ in which any symbol that occurs in a row or column, occurs exactly $d$ times in that row or column. We show how to construct a $d$-balanced equi-$n$-square from a partition of a Latin square of order $n$ into $d \times (n/d)$ subrectangles. In design theory, $L$ is equivalent to a decomposition of $K_{n,n}$ into $d$-regular spanning subgraphs of $K_{n/d,n/d}$. We also study when $L$ is diagonally cyclic, defined as when $l_{(i+1)(j+1)} = l_{ij} + 1$ for all $i, j \in \mathbb{Z}_n$, which corresponds to cyclic such decompositions of $K_{n,n}$ (and thus $\alpha$-labellings).

We identify necessary conditions for the existence of (a) $d$-balanced equi-$n$-squares, (b) diagonally cyclic $d$-balanced equi-$n$-squares, and (c) Latin squares of order $n$ which partition into $d \times (n/d)$ subrectangles. We prove the necessary conditions are sufficient for arbitrary fixed $d \geq 1$ when $n$ is sufficiently large, and we resolve the existence problem completely when $d \in \{1, 2, 3\}$.

Along the way, we identify a bijection between $\alpha$-labellings of $d$-regular bipartite graphs and, what we call, $d$-starters: matrices with exactly one filled cell in each top-left-to-bottom-right unbroken diagonal, and either $d$ or 0 filled cells in each row and column. We use $d$-starters to construct diagonally cyclic $d$-balanced equi-$n$-squares, but this also gives new constructions of $\alpha$-labellings.