Strongly Deza graphs

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A Deza graph G with parameters (n, k, b, a) is a k-regular graph of order n for which the number of common neighbours of two vertices takes just two values, b or a, where $b \geqslant a$. Moreover, G is not the complete graph or the edgeless graph. Deza graphs were introduced in [3], and the name was given in [4], where the basics of Deza graph theory were founded and different constructions of Deza graphs were presented. Strongly regular graphs are a particular case of Deza graphs.

Deza graphs can be considered in terms of matrices. Let G be a graph with n vertices, and M be its adjacency matrix. Then G is a Deza graph with parameters (n, k, b, a) if and only if

$$M^2 = aA + bB + kI$$

for some symmetric $(0, 1)$-matrices A and B such that $A + B + I = J$, where J is the all ones matrix and I is the identity matrix. Graphs G_A and G_B with matrices A and B are called the children of G.

Definition. *A Deza graph is called a strongly Deza graph if its children are strongly regular graphs.*

Theorem 1. [1, Theorem 3.2] *Let G be a Deza graph with parameters (n, k, b, a), $b > a$. Let M, A, B be the adjacency matrices of G and its children, respectively. If $\theta_1 = k, \theta_2, \dots, \theta_n$ are the eigenvalues of M, then*

(i) *the eigenvalues of A are*

$$\alpha = \frac{b(n-1) - k(k-1)}{b-a}, \ \alpha_2 = \frac{k - b - \theta_2^2}{b-a}, \ \dots, \ \alpha_n = \frac{k - b - \theta_n^2}{b-a};$$

(ii) *the eigenvalues of B are*

$$\beta = \frac{a(n-1) - k(k-1)}{a-b}, \ \beta_2 = \frac{k - a - \theta_2^2}{a-b}, \ \dots, \ \beta_n = \frac{k - a - \theta_n^2}{a-b}.$$

By Theorem above, a strongly Deza graph has at most three distinct absolute values of its eigenvalues.

Theorem 2. *Suppose G is a strongly Deza graph with parameters (n, k, b, a). Then*

(i) *G has at most five distinct eigenvalues.*

(ii) *If G has two distinct eigenvalues, then $a = 0$, $b = k - 1 \geqslant 1$, and G is a disjoint union of cliques of order $k + 1$.*

(iii) *If G has three distinct eigenvalues, then G is a strongly regular graph with parameters (n, k, λ, μ), where $\{\lambda, \mu\} = \{a, b\}$, or G is disconnected and each component is a strongly regular graph with parameters (v, k, b, b), or each component is a complete bipartite graph $K_{k,k}$ with $k \geqslant 2$.*

If G is a bipartite graph, then the *halved* graphs of G are two connected components of the graph on the same vertex set, where two vertices are adjacent whenever they are at distance two in G.

The next theorem gives a spectral characterization of strongly Deza graphs.

Theorem 3. *Let G be a connected Deza graph with parameters (n, k, b, a), $b > a$, and it has at most three distinct absolute values of its eigenvalues.*

(i) *If G is a non-bipartite graph, then G is a strongly Deza graph.*

(ii) *If G is a bipartite graph, then either G is a strongly Deza graph or its halved graphs are strongly Deza graphs.*

We also discuss some results on distance-regular strongly Deza graphs.

The main results of the talk are presented in [2].

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