Strongly Deza graphs

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A Deza graph $G$ with parameters $(n, k, b, a)$ is a $k$-regular graph of order $n$ for which the number of common neighbours of two vertices takes just two values, $b$ or $a$, where $b \geq a$. Moreover, $G$ is not the complete graph or the edgeless graph. Deza graphs were introduced in [3], and the name was given in [4], where the basics of Deza graph theory were founded and different constructions of Deza graphs were presented. Strongly regular graphs are a particular case of Deza graphs.

Deza graphs can be considered in terms of matrices. Let $G$ be a graph with $n$ vertices, and $M$ be its adjacency matrix. Then $G$ is a Deza graph with parameters $(n, k, b, a)$ if and only if

$$M^2 = aA + bB + kI$$

for some symmetric $(0, 1)$-matrices $A$ and $B$ such that $A + B + I = J$, where $J$ is the all ones matrix and $I$ is the identity matrix. Graphs $G_A$ and $G_B$ with matrices $A$ and $B$ are called the children of $G$.

**Definition.** A Deza graph is called a strongly Deza graph if its children are strongly regular graphs.
Theorem 1. [1, Theorem 3.2] Let $G$ be a Deza graph with parameters $(n, k, b, a)$, $b > a$. Let $M, A, B$ be the adjacency matrices of $G$ and its children, respectively. If $\theta_1, \theta_2, \ldots, \theta_n$ are the eigenvalues of $M$, then

(i) the eigenvalues of $A$ are

$$\alpha = \frac{b(n - 1) - k(k - 1)}{b - a}, \quad \alpha_2 = \frac{k - b - \theta_2^2}{b - a}, \ldots, \quad \alpha_n = \frac{k - b - \theta_n^2}{b - a};$$

(ii) the eigenvalues of $B$ are

$$\beta = \frac{a(n - 1) - k(k - 1)}{a - b}, \quad \beta_2 = \frac{k - a - \theta_2^2}{a - b}, \ldots, \quad \beta_n = \frac{k - a - \theta_n^2}{a - b}.$$

By Theorem above, a strongly Deza graph has at most three distinct absolute values of its eigenvalues.

Theorem 2. Suppose $G$ is a strongly Deza graph with parameters $(n, k, b, a)$. Then

(i) $G$ has at most five distinct eigenvalues.

(ii) If $G$ has two distinct eigenvalues, then $a = 0$, $b = k - 1 \geq 1$, and $G$ is a disjoint union of cliques of order $k + 1$.

(iii) If $G$ has three distinct eigenvalues, then $G$ is a strongly regular graph with parameters $(n, k, \lambda, \mu)$, where $\{\lambda, \mu\} = \{a, b\}$, or $G$ is disconnected and each component is a strongly regular graph with parameters $(v, k, b, b)$, or each component is a complete bipartite graph $K_{k,k}$ with $k \geq 2$.

If $G$ is a bipartite graph, then the halved graphs of $G$ are two connected components of the graph on the same vertex set, where two vertices are adjacent whenever they are at distance two in $G$.

The next theorem gives a spectral characterization of strongly Deza graphs.

Theorem 3. Let $G$ be a connected Deza graph with parameters $(n, k, b, a)$, $b > a$, and it has at most three distinct absolute values of its eigenvalues.

(i) If $G$ is a non-bipartite graph, then $G$ is a strongly Deza graph.

(ii) If $G$ is a bipartite graph, then either $G$ is a strongly Deza graph or its halved graphs are strongly Deza graphs.

We also discuss some results on distance-regular strongly Deza graphs. The main results of the talk are presented in [2]. The work of the speaker is supported by the Mathematical Center in Akademgorodok, the agreement with Ministry of Science and High Education of the Russian Federation number 075-15-2019-1613.
References


