On the complex hypothesis of Banach

Luis Montejano

Instituto de Matematicas

luismontej@gmail.com

The following is known as the geometric hypothesis of Banach: let $V$ be an $m$-dimensional Banach space (over the real or the complex numbers) with unit ball $B$ and suppose all $n$-dimensional subspaces of $V$ are isometric (all the $n$-sections of $B$ are affinely equivalent). In 1932, Banach conjectured that under this hypothesis $V$ is a Hilbert space (the boundary of $B$ is an ellipsoid). Gromow proved in 1967 that the conjecture is true for $n$=even and Dvoretzky and V. Milman derived the same conclusion under the hypothesis $n$=infinity. We prove this conjecture for $n = 4k + 1$, with the possible exception of $V$ a real Banach space and $n = 133$. [G.Bor, L.Hernandez-Lamoneda, V. Jiménez and L. Montejano. To appear Geometry & Topology] for the real case and [J. Bracho, L. Montejano, submitted to J. of Convex Analysis] for the complex case.

The ingredients of the proof are classical homotopic theory, irreducible representations of the orthogonal group and convex geometry. For the complex case, suppose $B$ is a convex body contained in complex space $\mathbb{C}^{n+1}$, with the property that all its complex $n$-sections through the origin are complex affinity equivalent to a fixed complex $n$-dimensional body $K$. Studying the topology of the complex fibre bundle $SU(n)\rightarrow SU(n-1)\rightarrow S^{2n+1}$, it is possible to prove that if $n$=even, then $K$ must be a ball and using homotopical properties of the irreducible representations we prove that if $n = 4 + 1$ then $K$ must be a body of revolution. Finally, we prove, using convex geometry and topology that, if this is the case, then there must be a section of $B$ which is an complex ellipsoid and consequently $B$ must be also a complex ellipsoid.