On the complex hypothesis of Banach

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The following is known as the geometric hypothesis of Banach: let V be an m-dimensional Banach space (over the real or the complex numbers) with unit ball B and suppose all n-dimensional subspaces of V are isometric (all the n-sections of B are affinely equivalent). In 1932, Banach conjectured that under this hypothesis V is a Hilbert space (the boundary of B is an ellipsoid). Gromow proved in 1967 that the conjecture is true for n=even and Dvoretzky and V. Milman derived the same conclusion under the hypothesis n=infinity. We prove this conjecture for $n = 4k + 1$, with the possible exception of V a real Banach space and $n = 133$. [G.Bor, L.Hernandez-Lamoneda, V. Jiménez and L. Montejano. To appear Geometry & Topology] for the real case and [J. Bracho, L. Montejano, submitted to J. of Convex Analysis] for the complex case.

The ingredients of the proof are classical homotopic theory, irreducible representations of the orthogonal group and convex geometry. For the complex case, suppose B is a convex body contained in complex space C^{n+1}, with the property that all its complex n-sections through the origin are complex affinity equivalent to a fixed complex n-dimensional body K. Studying the topology of the complex fibre bundle $SU(n)- > SU(n-1)- > S^{2n+1}$, it is possible to prove that if n=even, then K must be a ball and using homotopical properties of the irreducible representations we prove that if $n = 4 + 1$ then K must be a body of revolution. Finally, we prove, using convex geometry and topology that, if this is the case, then there must be a section of B which is an complex ellipsoid and consequently B must be also a complex ellipsoid.