Characterization of manifolds of constant curvature by spherical curves and ruled surfaces

Luiz C. B. Da Silva
Weizmann Institute of Science
luiz.da-silva@weizmann.ac.il

José D. Da Silva
Federal Rural University of Pernambuco
jose.dsilva@ufrpe.br

Space forms, i.e., Riemannian manifolds of constant sectional curvature, play a prominent role in geometry and an important problem consists of finding properties that characterize them. In this talk, we report results from [1], where we show that the validity of some theorems concerning curves and surfaces can be used for this purpose. For example, it is known that the so-called rotation minimizing (RM) frames allow for a characterization of geodesic spherical curves in Euclidean, hyperbolic, and spherical spaces through a linear equation involving the coefficients that dictate the RM frame motion [2]. Here, we shall prove the converse, i.e., if all geodesic spherical curves on a manifold are characterized by a certain linear equation, then all the geodesic spheres with a sufficiently small radius are totally umbilical, and consequently, the ambient manifold is a space form. (We also present an alternative proof, in terms of RM frames, for space forms as the only manifolds where all geodesic spheres are totally umbilical [3].) In addition, we furnish two other characterizations in terms of (i) an inequality involving the mean curvature of a geodesic sphere and the curvature function of their curves and (ii) the vanishing of the total torsion of closed spherical curves in the case of 3d manifolds. (These are the converse of previous results [4].) Finally, we introduce ruled surfaces and show that if all extrinsically flat surfaces in a 3d manifold are ruled, then the manifold is a space form.

[1] Da Silva, L.C.B. and Da Silva, J.D.: "Characterization of manifolds of constant curvature by spherical curves". Annali di Matematica **199**, 217 (2020); Da Silva, L.C.B. and Da Silva, J.D.: "Ruled and extrinsically flat surfaces in three-dimensional manifolds of constant curvature". Unpublished manuscript 2021.
[2] Bishop, R.L.: "There is more than one way to frame a curve". Am. Math. Mon. **82**, 246 (1975); Da Silva, L.C.B. and Da Silva, J.D.: "Characterization of curves that lie on a geodesic sphere or on a totally geodesic hypersurface in a hyperbolic space or in a sphere". Mediterr. J. Math. **15**, 70 (2018)

[3] Kulkarni, R.S.: "A finite version of Schur's theorem". Proc. Am. Math. Soc. **53**, 440 (1975); Vanhecke, L. and Willmore, T.J.: "Jacobi fields and geodesic spheres". Proc. R. Soc. Edinb. A **82**, 233 (1979); Chen, B.Y. and Vanhecke, L.: "Differential geometry of geodesic spheres". J. Reine Angew. Math. **325**, 28 (1981).
[4] Baek, J., Kim, D.S., and Kim, Y.H.: "A characterization of the unit sphere". Am. Math. Mon. **110**, 830 (2003); Pansonato, C.C. and Costa, S.I.R., "Total torsion of curves in three-dimensional manifolds". Geom. Dedicata **136**, 111 (2008).