

**Blow-up phenomena in nonlocal eigenvalue problems:  
when theories of  $L^1$  and  $L^2$  meet**Hardy Chan*ETH Zürich*

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We develop a linear theory of very weak solutions for nonlocal eigenvalue problems  $\mathcal{L}u = \lambda u + f$  involving integro-differential operators posed in bounded domains with homogeneous Dirichlet exterior condition, with and without singular boundary data. We consider mild hypotheses on the Green's function and the standard eigenbasis of the operator. The main examples in mind are the fractional Laplacian operators.

Without singular boundary datum and when  $\lambda$  is not an eigenvalue of the operator, we construct an  $L^2$ -projected theory of solutions, which we extend to the optimal space of data for the operator  $\mathcal{L}$ . We present a Fredholm alternative as  $\lambda$  tends to the eigenspace and characterise the possible blow-up limit. The main new ingredient is the transfer of orthogonality to the test function.

We then extend the results to singular boundary data and study the so-called large solutions, which blow up at the boundary. For that problem we show that, for any regular value  $\lambda$ , there exist “large eigenfunctions” that are singular on the boundary and regular inside. We are also able to present a Fredholm alternative in this setting, as  $\lambda$  approaches the values of the spectrum.

We also obtain a maximum principle for weighted  $L^1$  solutions when the operator is  $L^2$ -positive. It yields a global blow-up phenomenon as the first eigenvalue is approached from below.

Finally, we recover the classical Dirichlet problem as the fractional exponent approaches one under mild assumptions on the Green's functions. Thus “large eigenfunctions” represent a purely nonlocal phenomenon.