Singularity preserving maps on matrix algebras

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The first result on linear preservers was obtained by Ferdinand Georg Frobenius, who characterized linear maps on complex matrix algebra preserving the determinant.

Let $M_n(\mathbb{F})$ be the $n \times n$ matrix algebra over a field $\mathbb{F}$ and $\mathcal{Y}$ be a subset of $M_n(\mathbb{F})$. We say that a transformation $T : \mathcal{Y} \rightarrow M_n(\mathbb{F})$ is of a standard form if there exist non-singular matrices $P, Q$ such that

\[
T(A) = PAQ \quad \text{or} \quad T(A) = PA^TQ \quad \text{for all} \ A \in \mathcal{Y}.
\] (1)

Frobenius [1] proved that if $T : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is linear and preserves the determinant, i.e., $\det(T(A)) = \det(A)$ for all $A \in M_n(\mathbb{C})$, then $T$ is of the standard form (1) with $\det(PQ) = 1$. In 1949 Jean Dieudonné [2] generalized this result for an arbitrary field $\mathbb{F}$. He replaced the determinant preserving condition by the singularity preserving condition and proved the corresponding result for a bijective map $T$.

In 2002 Gregor Dolinar and Peter Šemrl [3] modified the classical result of Frobenius by removing the linearity and replacing the determinant preserving condition by

\[
\det(A + \lambda B) = \det(T(A) + \lambda T(B)) \quad \text{for all} \ A, B \in M_n(\mathbb{F}) \ \text{and all} \ \lambda \in \mathbb{F} \quad (2)
\]

for $\mathbb{F} = \mathbb{C}$. They proved that if $T : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is surjective and satisfies (2), then $T$ is linear and hence is of the standard form (1) with $\det(PQ) = 1$.

Soon after that, Victor Tan and Fei Wang [4] generalized this proof for a field $\mathbb{F}$ with $|\mathbb{F}| > n$ and showed that under the condition (2) the map $T$ is linear even without the surjectivity condition. Moreover, they revealed that if $T$ is surjective, then only two different values of $\lambda$ are required in (2).
be more precise, if $|\mathbb{F}| > n$ and $T: M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a surjective map satisfying

$$
\det(A + \lambda_i B) = \det(T(A) + \lambda_i T(B)) \quad \text{for all } A, B \in M_n(\mathbb{F}) \text{ and } i = 1, 2,
$$

where $\lambda_i \neq 0$ and $(\lambda_1/\lambda_2)^k \neq 1$ for $1 \leq k \leq n - 2$, then $T$ is of the standard form (1).

Nevertheless, this result was also further generalized by Constantin Costara [5]. Suppose $|\mathbb{F}| > n^2$ and $\lambda_0 \in \mathbb{F}$. Let $T: M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a surjective map satisfying (2) only for one fixed value of $\lambda = \lambda_0$: $\det(A + \lambda_0 B) = \det(T(A) + \lambda_0 T(B))$ for all $A, B \in M_n(\mathbb{F})$.

Costara obtained that if $\lambda_0 \neq -1$, then such $T$ is of the standard form (1) with $\det(PQ) = 1$. For $\lambda_0 = -1$, he showed that there exist $P, Q \in GL_n(\mathbb{F})$, $\det(PQ) = 1$, and $A_0 \in M_n(\mathbb{F})$ such that

$$
T(A) = P(A + A_0)Q \quad \text{or} \quad T(A) = P(A + A_0)^TQ \quad \text{for all } A \in \mathcal{Y}.
$$

The aim of this work is to relax the condition (2) for $T$. It has been revealed that if $\mathbb{F}$ is an algebraically closed field, then the conditions on determinant in the above results of Dieudonné or Tan and Wang can be replaced by less restrictive. The following result has been obtained.

**Theorem.** Suppose $\mathcal{Y} = GL_n(\mathbb{F})$ or $\mathcal{Y} = M_n(\mathbb{F})$, $T: \mathcal{Y} \to M_n(\mathbb{F})$ is a map satisfying the following conditions:

- for all $A, B \in \mathcal{Y}$ and $\lambda \in \mathbb{F}$, the singularity of $A + \lambda B$ implies the singularity of $T(A) + \lambda T(B)$;

- the image of $T$ contains at least one non-singular matrix.

Then $T$ is of the standard form (1).

(Note that in the theorem above $\det(PQ)$ possibly differs from 1.)

**References**


