# Geometric Valuation Theory

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Valuations on compact convex sets in $\mathbb{R}^n$ play an active and prominent role in geometry. They were critical in Dehn's solution to Hilbert's Third Problem in 1901. They are defined as follows. A function Z whose domain is a collection of sets $\mathcal{S}$ and whose co-domain is an Abelian semigroup is called a *valuation* if

$$\mathrm{Z}(K) + \mathrm{Z}(L) = \mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L),$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{S}$.

The first classification result for valuations on the space of compact convex sets, $\mathcal{K}^n$, in $\mathbb{R}^n$ (where $\mathcal{K}^n$ is equipped with the topology induced by the Hausdorff metric) was established by Blaschke.

THEOREM. **(Blaschke)** *A functional* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous, translation and* $SL(n)$ *invariant valuation if and only if there are* $c_0, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + c_n V_n(K)$$

*for every* $K \in \mathcal{K}^n$.

Probably the most famous result in the geometric theory of valuations is the Hadwiger characterization theorem.

THEOREM. **(Hadwiger)** *A functional* $\mathrm{Z} : \mathcal{K}^n \to \mathbb{R}$ *is a continuous and rigid motion invariant valuation if and only if there are* $c_0, \dots, c_n \in \mathbb{R}$ *such that*

$$\mathrm{Z}(K) = c_0 V_0(K) + \cdots + c_n V_n(K)$$

*for every* $K \in \mathcal{K}^n$.

Here $V_0(K), \dots, V_n(K)$ are the intrinsic volumes of $K \in \mathcal{K}^n$. In particular, $V_0(K)$ is the Euler characteristic of $K$, while $2\,V_{n-1}(K)$ is the surface are of $K$ and $V_n(K)$ the $n$-dimensional volume of $K$. Hadwiger's theorem shows that the intrinsic volumes are the most basic functionals in Euclidean geometry. It finds powerful applications in Integral Geometry and Geometric Probability.

The fundamental results of Blaschke and Hadwiger have been the starting point of the development of Geometric Valuation Theory. Classification results for valuations invariant (or covariant) with respect to important groups are central questions. The talk will give an overview of such results including recent extensions to valuations on function spaces.

# References

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