Exponential dichotomy conditions for difference equations with perturbed coefficients

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A system of linear difference equations with periodic coefficients is considered

\[ y_{n+1} = (A(n) + B(n))y_n, \quad n \in \mathbb{Z}, \quad (1) \]

where \( A(n) \) are non-degenerate matrices of size \( m \times m \) and the matrix sequence \( \{A(n)\} \) is \( N \)-periodic, i.e. \( A(n + N) = A(n), \quad n \in \mathbb{Z} \). The sequence \( \{B(n)\} \) is an \( N \)-periodic sequence of perturbations. We assume that the system

\[ x_{n+1} = A(n)x_n, \quad n \in \mathbb{Z}, \quad (2) \]

is exponentially dichotomous. As shown in [1], this is equivalent to the fact that there are Hermitian matrices \( H(0), H(1), \ldots, H(N-1) \) and a matrix \( P \) satisfying the following boundary value problem

\[
\begin{cases}
H(l) - A^*(l)H(l + 1)A(l) = \left( U_l^* \right)^{-1} P^* U_l^* U_l PU_l^{-1} \\
- \left( U_l^* \right)^{-1} (I - P)^* U_l^* U_l (I - P) U_l^{-1}, \quad l = 0, 1, \ldots, N - 1, \\
H(0) = H(N) > 0, \\
H(0) = P^* H(0) P + (I - P)^* H(0) (I - P), \\
P^2 = P, \quad PU_N = U_N P,
\end{cases}
\]

where \( U_l \) is the Cauchy matrix of (2). This criterion is analogous to the criterion of M. G. Krein for the exponential dichotomy of difference equations with constant coefficients [2].

Using the fact that the solution of the boundary value problem (3) is represented as

\[
H(l) = \left( U_l^* \right)^{-1} \left( \sum_{k=0}^{\infty} \left( U_N^k \right)^{k} P^* \left( \sum_{i=l}^{N+l-1} U_i^* U_i \right) PU_N^k \right) U_l^{-1} \\
+ \left( U_l^* \right)^{-1} \left( \sum_{k=1}^{\infty} \left( U_N^k \right)^{k} (I - P)^* \left( \sum_{i=l}^{N+l-1} U_i^* U_i \right) (I - P) U_N^k \right) U_l^{-1} = H^-(l) + H^+(l),
\]

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we can obtain conditions for perturbations \{B(n)\} under which the system (1) is also exponentially dichotomous.

**Theorem.** Let \( \det(A(n)) \neq 0 \) and the matrix sequence of perturbations \{B(n)\} satisfy the condition

\[
\max\{\|B(0)\|, \ldots, \|B(N-1)\|\} < \left( h^- \left( \sqrt{1 - \frac{1}{h^-}} + 1 \right) \sqrt{h^- \|H(0)\|} + h^+ \left( \sqrt{1 + \frac{1}{h^+}} + 1 \right) \sqrt{h^+ \|H(0)\|} \right)^{-1},
\]

where

\[
h^- = \max\{\|H^-(0)\|, \|H^-(1)\|, \ldots, \|H^-(N-1)\|\},
\]

\[
h^+ = \max\{\|H^+(0)\|, \|H^+(1)\|, \ldots, \|H^+(N-1)\|\},
\]

then the perturbed system (1) is exponentially dichotomous.

This paper is a continuation of [1, 3–5].

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**Literature**


