Closures of solvable permutation groups

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Let \( m \) be a positive integer and let \( \Omega \) be a finite set. The \( m \)-closure \( G^{(m)} \) of \( G \leq \text{Sym}(\Omega) \) is the largest permutation group on \( \Omega \) having the same orbits as \( G \) in its induced action on the Cartesian product \( \Omega^m \). Wielandt [5] showed that

\[
G^{(1)} \geq G^{(2)} \geq \cdots \geq G^{(m)} = G^{(m+1)} = \cdots = G,
\]

for some \( m < |\Omega| \). (Since the stabilizer in \( G \) of all but one point is always trivial, \( G^{(n-1)} = G \) where \( n = |\Omega| \).) In this sense, the \( m \)-closure can be considered as a natural approximation of \( G \). It was shown by Praeger and Saxl [2] that for \( m \geq 6 \), the \( m \)-closure \( G^{(m)} \) of a primitive permutation group \( G \) has the same socle as \( G \). Furthermore, they classified explicitly primitive groups \( G \) and \( H \) with different socles having the same \( m \)-orbits for \( m \leq 5 \). Unfortunately, their results say very little about closures of solvable permutation groups. The main goal of this talk is to present the results of [1], where we study such closures.

The 1-closure of \( G \) is the direct product of symmetric groups \( \text{Sym}(\Delta) \), where \( \Delta \) runs over the orbits of \( G \). Thus the 1-closure of a solvable group is solvable if and only if each of its orbits has cardinality at most 4. The case of 2-closure is more interesting. The 2-closure of every (solvable) 2-transitive group \( G \leq \text{Sym}(\Omega) \) is \( \text{Sym}(\Omega) \); other examples of solvable \( G \) and nonsolvable \( G^{(2)} \) appear in [4]. But, as shown by Wielandt [5], each of the classes of finite \( p \)-groups and groups of odd order is closed with respect to taking the 2-closure. Currently, no characterization of solvable groups having solvable 2-closure is known.
Seress [3] observed that if $G$ is a primitive solvable group, then $G^{(5)} = G$; so the 5-closure of a primitive solvable group is solvable. Our main result is the following stronger statement.

**Theorem 1.** The 3-closure of a solvable permutation group is solvable.

The corollary below is an immediate consequence of Theorem 1 and the chain of inclusions (1).

**Corollary 2.** For every integer $m \geq 3$, the $m$-closure of a solvable permutation group is solvable.

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**References**


