Some observations about regular maps

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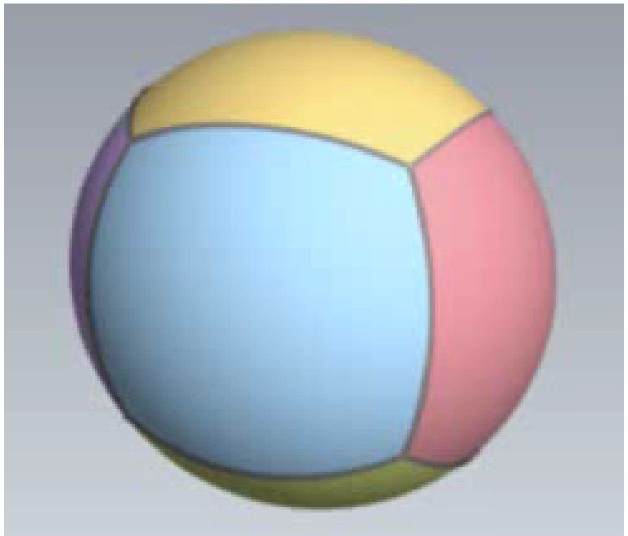
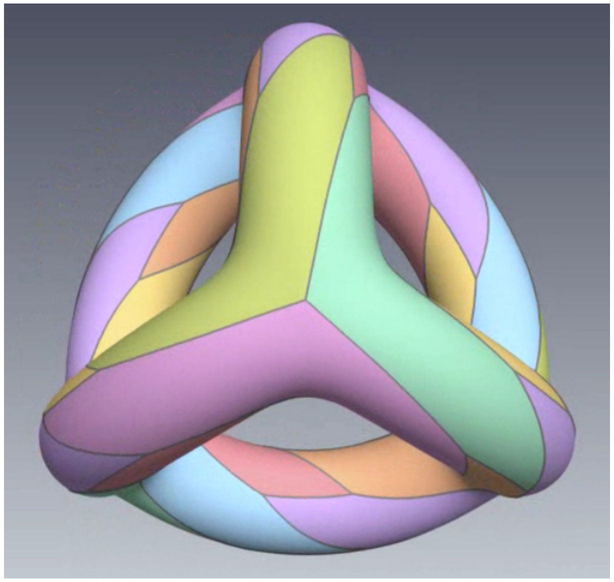
— including some joint work with various others

Regular maps

Regular maps are symmetric embeddings of (connected) graphs or multigraphs on surfaces:



Q_3 on sphere



Klein map (genus 3)

Some definitions

A map map is a 2-cell embedding of a connected graph or multigraph on a surface, and an automorphism of such a map M is a bijection from M to M preserving the sets of vertices, edges and faces and incidence among them.

Every automorphism of a map M is uniquely determined by its effect on a given flag (incident vertex-edge-face triple), and it follows that $|\text{Aut}\, M| \leq 4|E|$ where E is the edge set.

A map M is (fully) regular if Aut M is transitive on flags, and orientably-regular if the carrier surface is orientable and $\text{Aut}^+ M$ is transitive on arcs (incident vertex-edge pairs).

An orientably-regular map is either 'reflexible' or 'chiral', depending on whether or not it admits reflections – e.g. fixing an arc but swapping the two faces incident with it.

Some history

- Examples were studied by Dyck, Klein & Heffter (1800s)

- Theory of symmetric maps was initiated by Brahana in the 1920s, and has been developed since then by various people such as Coxeter, Wilson, Jones and Singerman

- Macbeath (1960s) laid the foundations for finding all orientably-regular or regular maps with automorphism group containing some $\mathrm{PSL}(2, q)$ as a subgroup of index 1 or 2

- Connections with Algebraic Geometry and Galois Theory [Belyi (1979), Grothendieck (1997), Jones et al (2007)]: the absolute Galois group can be studied via actions on maps

- Also some tentative applications to structural chemistry (fullerenes and nanotubes, etc.).

Transitivity, type and genus

If M is a regular or orientably-regular map, then $\mathrm{Aut}\, M$ is transitive on vertices, edges, arcs and faces.

In particular, every face of must have the same number of edges (say m) and every vertex must have the same valency (say k). In this case we say that M has type $\{m, k\}$. Also if M is regular, with $|V|$ vertices, $|E|$ edges and $|F|$ faces, then the characteristic χ and genus g of the carrier surface is given by the Euler-Poincaré formula

$$\chi = |V| - |E| + |F| = |\mathrm{Aut}\, M| \left(\frac{1}{2k} - \frac{1}{4} + \frac{1}{2m} \right)$$

$$= \begin{cases} 2 - 2g & \text{if } M \text{ is orientable} \\ 2 - g & \text{if } M \text{ is non-orientable.} \end{cases}$$

A similar formula holds in the orientably-regular case.

Connection with triangle groups (for type $\{m, k\}$)

If M is orientably-regular, then for any flag (v, e, f) there exist orientation-preserving automorphisms R and S s.t.

- R induces a single-step local rotation of order m about f
- S induces a single-step local rotation of order k about v
- RS induces a local rotation of order 2 about e.

By connectedness, these generate the group $\mathrm{Aut}^{+}(M)$ of orientation-preserving automorphisms, and satisfy the ordinary $(2, k, m)$ triangle group relations $R^m = S^k = (RS)^2 = 1$.

Conversely, every such M of type $\{m, k\}$ can be constructed from a 'smooth' finite quotient of the latter group.

Similarly, every fully regular M of type $\{k, m\}$ can be constructed from a 'smooth' finite quotient of the extended/full $(2, k, m)$ triangle group

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ab)^2 = 1 \,\rangle.$$

This means that the construction and analysis of regular and orientably-regular maps can be reduced to group theory!

This enabled a computer-aided determination [MC,2012] of

- all orientably-regular maps of genus up to 301, and
- all non-orientable fully regular maps of genus up to 602.

Totals: 43144 orientable (29246 reflexible, 13898 chiral), and 6386 non-orientable, counting duals and mirror-images when they are not isomorphic to others on the lists.

Some unexpected theoretical discoveries helped by patterns in the computational data

- New infinitely families of examples of regular/chiral maps
- Symmetric embeddings of given families of graphs
- Infinitely many gaps in genus spectra of some families:
  - ⋆ non-orientable regular maps
  - ⋆ orientably-regular but chiral maps
  - ⋆ fully regular maps with simple underlying graph
- Reg. maps admitting Coxeter/Wilson operators
- Consequences for 'large' group actions on surfaces
- Consequences for regular or chiral polyhedra/polytopes

... plus others [127 citations of two papers on the data].

Some open questions about regular maps

- What kinds of groups are the most prevalent as the group of orientation-preserving automorphisms? (Simple groups? insoluble groups? soluble groups? 2-groups?)

- What types $\{m, k\}$ occur the most frequently among orientably-regular maps on hyperbolic surfaces (genus > 1)?

- Big one: Among orientably-regular maps of genus > 0, which of reflexibility and chirality is the most prevalent?

- Which of reflexibility and chirality is the most prevalent for a given type $\{m, k\}$?

What kinds of groups are the most prevalent?

A lot of research has concentrated on orientably-regular maps with simple or almost-simple automorphism groups:

- $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ [Macbeath (1960s)]
- $\mathrm{Alt}(n)$ and $\mathrm{Sym}(n)$ [Higman, MC & Everitt (1970s–90s)]
- Sporadic finite simple groups [G. Jones & others]
- Other simple groups of Lie type [M. Liebeck & others]

This might have left an impression that such maps are prevalent. But they are not! Well, at least not for small genus:

- Genus 0: $\mathrm{Aut}\, M$ is soluble for all M except two
- Genus 1: $\mathrm{Aut}\, M$ is soluble for all M
- Genus 2 to 301: $\mathrm{Aut}\, M$ is soluble in over 92% of cases

Some of these numbers might surprise you!

Orientation-preserving subgroups of orientably-regular maps of genus 2 to 301

	Reflexible	Chiral	Combined
Total	29246	13898	43144
Non-abelian simple	159	0	159
Cyclic	900	0	900
Abelian	1200	0	1200
2-group	2474	618	3092
Insoluble	2870	384	3254
Nilpotent	5028	654	5682
Soluble	26376	13514	39890

Question: How does this pattern change for higher genera?

What types $\{m, k\}$ occur the most frequently?

- Genus 0: one each of types $\{2, k\}$, $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$
  ... plus duals of types $\{k, 2\}$, $\{4, 3\}$ and $\{5, 3\}$
- Genus 1: infinitely many of each of types $\{3, 6\}$ and $\{4, 4\}$
  ... plus duals of types $\{6, 3\}$
- Genus 2 to 301: Guesses?

What types $\{m, k\}$ occur the most frequently? (cont.)

- Genus 0: one each of types $\{2, k\}$, $\{3, 3\}$, $\{3, 4\}$, $\{3, 5\}$
  ... plus duals of types $\{k, 2\}$, $\{4, 3\}$ and $\{5, 3\}$
- Genus 1: infinitely many of each of types $\{3, 6\}$ and $\{4, 4\}$
  ... plus duals of types $\{6, 3\}$
- Genus 2 to 301:

	Reflexible	Chiral	Combined
Type $\{8, 8\}$	551	896	1447
Type $\{4, 8\}$ or $\{8, 4\}$	426	860	1286
Type $\{6, 12\}$ or $\{12, 6\}$	640	616	1256
Type $\{6, 6\}$	344	830	1174
Type $\{12, 12\}$	534	634	1168
Type $\{8, 24\}$ or $\{24, 8\}$	696	280	976
Type $\{4, 12\}$ or $\{12, 4\}$	446	508	954
Type $\{12, 24\}$ or $\{24, 12\}$	642	272	914
Type $\{8, 12\}$ or $\{12, 8\}$	594	312	906

Most prevalent: Reflexibility? or chirality?

To me this is by far the most interesting question, but also quite a challenging question to fully answer.

For genus 0, the answer is quite easy: every orientable-regular map on the sphere is reflexible.

For genus 1, chirality dominates. This follows from the structure of the ordinary $(2,3,6)$, $(2,4,4)$ and $(2,6,3)$ triangle groups: each is an extension of an abelian normal subgroup $N \cong \mathbb{Z} \oplus \mathbb{Z}$ by a cyclic group (of order 6, 4 or 6), and factoring out a normal subgroup $K \leq N$ gives the automorphism group of a chiral map far more often than not (as K must be normal in the full triangle group for reflexibility).

Most prevalent: Reflexibility? or chirality? (cont.)

In principle, the same should hold for genus > 1. If Δ is the ordinary $(2, k, m)$ triangle group, and $\Delta/K \cong \text{Aut}^+M$, then M is reflexible if and only if K is normal in the full $(2, k, m)$ triangle group Δ^*. Remarkably, the latter happens more often than we might expect, especially for small genus:

Genus 2 to 101: 6104 reflexible, 2084 chiral (25.5% chiral)
Genus 102 to 201: 10427 reflexible, 4960 chiral (32.2%)
Genus 202 to 301: 12715 reflexible, 6854 chiral (35.0%)

Note that the percentage of chiral maps is increasing.

What happens for even higher genera?

Some ideas/suggestions

- Investigate the situation for soluble quotients Δ/K
- Do alternating/symmetric quotients eventually dominate?
- Study the situation for the most prevalent types $\{m, k\}$ especially those for which finite quotients of the ordinary $(2, k, m)$ triangle group have the highest density among finite groups generated by an involution and one other element
- Alternative group density considerations? (e.g. frequency of large characteristic cyclic subgroups of $\mathrm{Aut}^+ M$?)
- Find out what happens for chiral covers of a fully regular map M (by studying the effect of the generators of the full triangle group Δ^* on the fundamental group K)

Answers very welcome! (Or time to work on them)

THANK YOU