Covering skew lattices

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1. Introduction: motivation, basic notions
2. Voltages on lattices
3. Properties of lattices preserved by voltage constructions
4. Symmetric lattices
5. Conclusion and references

Goal

The goal of this talk is

1) to present
some interesting constructions of "covering" lattices, and

2) to show
how the concepts of "cover", "voltage graphs" and "symmetry",
apply to lattices and non-commutative lattices.
reveal unexpected connections between lattices and groups.

Motivation

KNOWN

- The concept of **covering graphs**, especially the technique of **voltage graphs** (Gross, Tucker 1974), is a useful tool for studying graphs, because
- it reduces the study of a bigger "covering structure" to the study of a smaller "base structure";

NEW

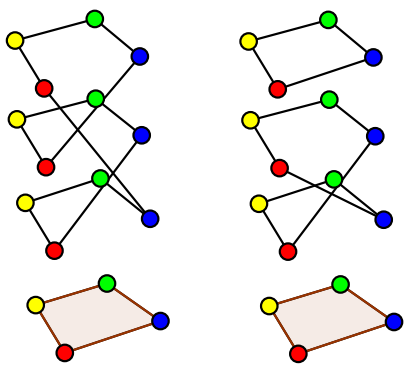
- since **lattices** may be regarded as **partially ordered sets** and hence represented by **directed graphs**, it is natural to ask:
- is it possible to apply this tool to lattices (and skew lattices, too), and if so, how to do it?

Covering graphs

- In short, **a covering graph** Γ' has locally the same structure as the smaller **base graph** Γ: there is a "natural projection" (a local homeomorphism) $p : \Gamma' \to \Gamma$.
- Such "covers" may be either "**regular**" or "**irregular**".

Example

Figure left: regular cover, figure right: irregular cover.



- There are several **covering graphs** constructions; the most elegant of them is the technique of **voltage graphs**.

Voltage graphs $< \Gamma, \alpha >$ and derived graphs Γ^{α}

Definition

Let Γ *be a* **directed graph** *and* G *a group.*
Let $\alpha : E(\Gamma) \to G$ *be the chosen* **assignment of voltages**.

The (right) **derived graph** Γ^{α} *of the* **base graph** $< \Gamma, \alpha >$, *called also the* **voltage graph**, *has the vertex set* $V(\Gamma^{\alpha}) = V(\Gamma) \times G$ *and the edge set* $E(\Gamma^{\alpha}) = E(\Gamma) \times G$.

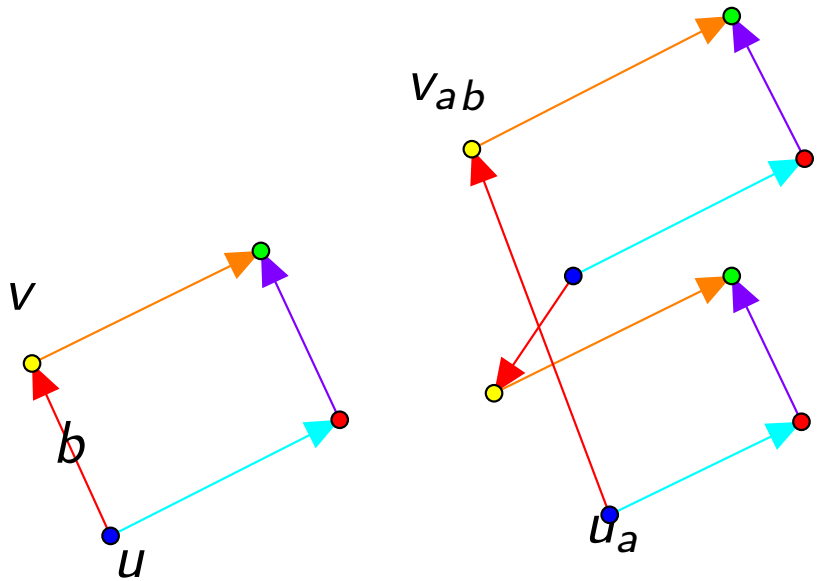
The set $\{v_a, a \in G\} \subset V(\Gamma^{\alpha})$ *is the* **fiber over** $v \in V(\Gamma)$ *and the set* $\{e_a, a \in G\} \subset E(\Gamma^{\alpha})$ *is the* **fiber over** $e \in E(\Gamma)$.

A directed edge $e = (u, v) = u \to v \in E(\Gamma)$ *with an assigned voltage* $b \in G$ *"lifts" to the fiber* $\{e_a, a \in G\} = \{(u_a, v_{ab}) = u_a \to v_{ab}, a \in G\}$ *in* Γ^{α}.

The opposite edges (u, v) *and* (v, u) *must have the inverse voltages* b *and* b^{-1}.

Example

In this "double cover" the only "non-zero voltage" is $b = 1 \in \mathbb{Z}_2$



Net voltages of paths and cycles

Definition

The **net voltage of a path** $a \to b \to \ldots m$ *is just the product of the voltages on these directed edges.*

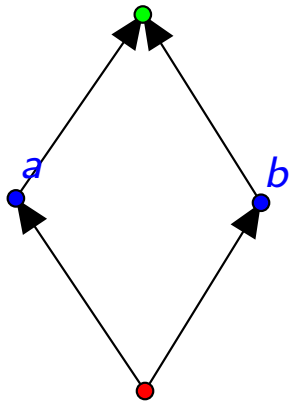
The **net voltage of a cycle** may depend on the choice of the initial vertex, but its **order** is independent of this choice.

Axioms of lattices $(L, \wedge, \vee)$

Definition

A **lattice** [2] is a set L with with two binary operations $\wedge$ and $\vee$ called **meet** and **join** satisfying pairs of idempotent, associative, absorption and commutative identities (axioms):

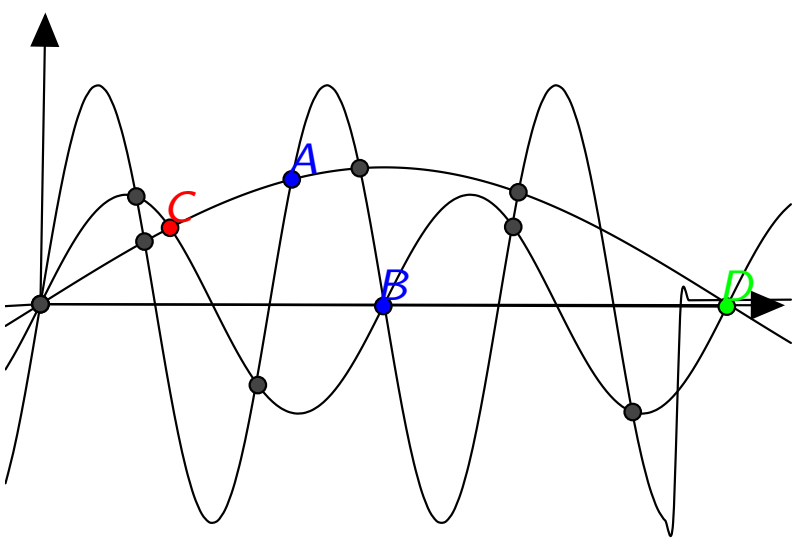
$$a \wedge a = a \qquad a \vee a = a$$
$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \qquad a \vee (b \vee c) = (a \vee b) \vee c$$
$$a \wedge (a \vee b) = a \qquad a \vee (a \wedge b) = b$$
$$a \wedge b = b \wedge a \qquad a \vee b = b \vee a$$

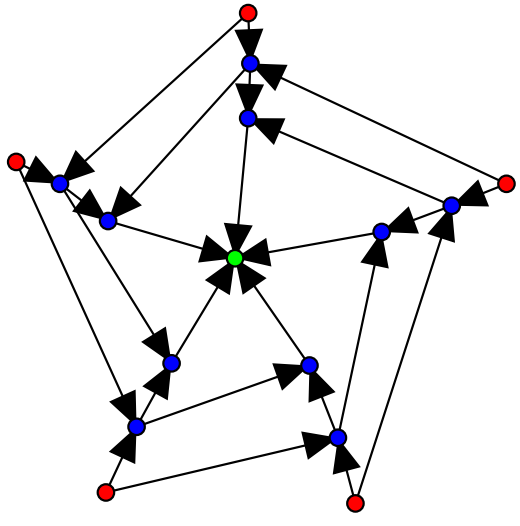


It is easy to obtain examples of lattices – just by "drawing directed planar curves" with the same initial and terminal point and taking all their "intersection points" as "points of L".

Example

In "function lattices", obtained from real-valued functions f_i defined on some interval $I \subset \mathbb{R}$, for any pair of points $A(c, f_{i(c)})$ and $B(d, f_j(d))$ their "join" $C = A \vee B$ and "meet" $D = A \wedge B$ are defined as the "closest intersection point on their right and left".





Noncommutative lattices, skew lattice

In a **noncommutative lattice** only the idempotent, associative and absorbtion identites hold.

The noncommutative lattices studied most extensively in recent years are **skew lattices** [2] characterized by identities

$$x \vee y = x \text{ if and only if } x \wedge y = y$$
$$x \vee y = y \text{ if and only if } x \wedge y = x.$$

Lattices as posets $(L, \geq)$

The lattice axioms define a **partial order** on L

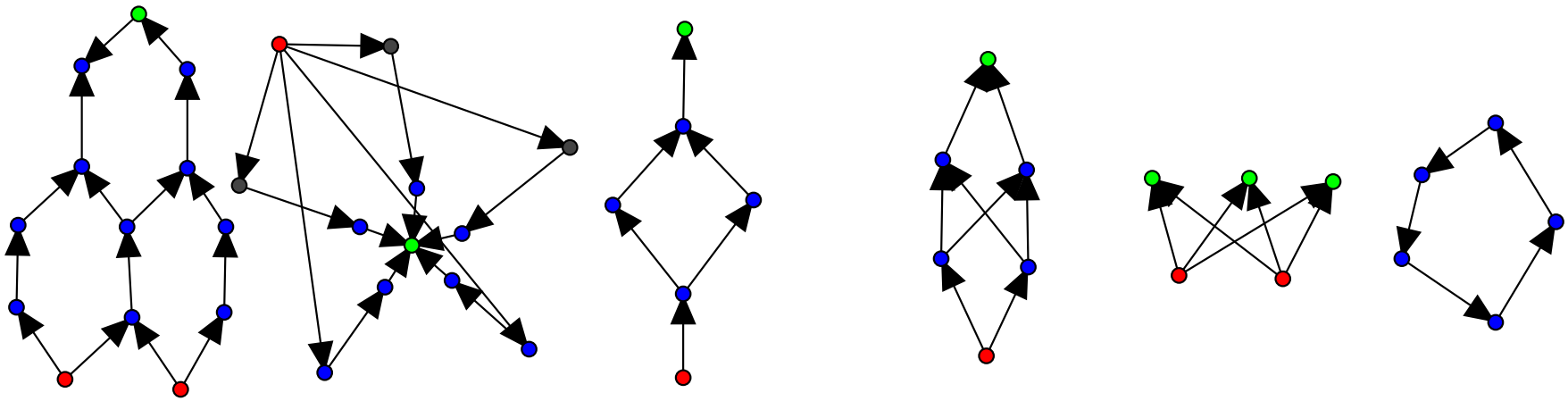
$$x \geq y \text{ if and only if } x \wedge y = y, \text{ or equivalently, } x \vee y = x.$$

Lattices may be thus represented by **partially ordered sets** $(L, \geq)$ and hence by **directed graphs**.

But it is not true that every directed graph represents a lattice.

Example

Which of the following directed graphs "are" (represent) lattices?

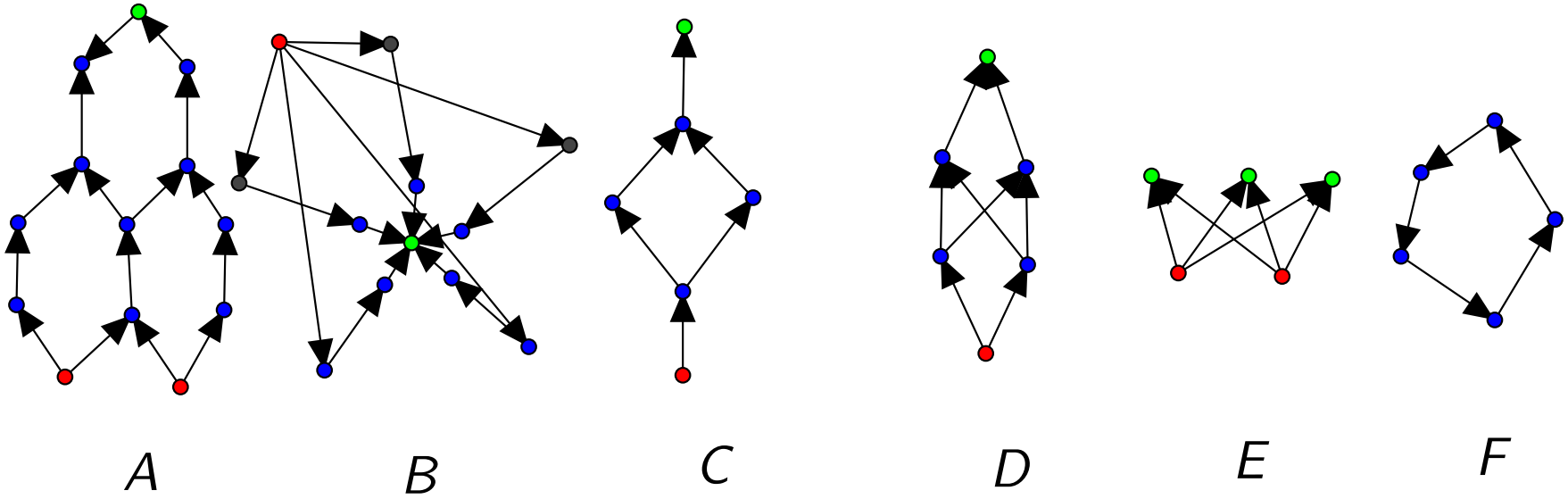


A B C D E F

Hint:

A directed graph Γ does not represent a lattice, if:

- it is not connected
- it contains a directed cycle $a \to b \to c \cdots \to a$
- it contains none or more than one maximal lower bound or none or more than one minimal upper bound of some $\{a, b\}$.



$a \sqcup b$ and $a \sqcap b$ – generalizations of $a \vee b$ and $a \wedge b$

Definition

Let P be a poset. For any $a, b \in P$ and for any $A \subset P$ let us define the following subsets of P:

- $A^{\uparrow} = \{x \in P, x \geq a, \forall a \in A\}$ upper bounds
- $A^{\downarrow} = \{x \in P, x \leq a, \forall a \in A\}$ lower bounds
- $A^{max} = \{x \in A, x \geq a, \forall a \in A\}$ maximal elements
- $A^{min} = \{x \in P, x \leq a, \forall a \in A\}$ minimal elements
- $a \sqcup b = (\{a, b\}^{\uparrow})^{min}$ **minimal upper bounds of** $\{a, b\}$
- $a \sqcap b = (\{a, b\}^{\downarrow})^{max}$ **maximal lower bounds of** $\{a, b\}$

Which posets represent lattices?

Characterization of posets representing lattices.

Lattices are exactly those posets $(L, \geq)$ for which the cardinalities of all these sets are one:

$$| a \sqcup b | = | a \sqcap b | = 1 \text{ for any } a, b, \in L$$

(and which satisfy the associativity condition).

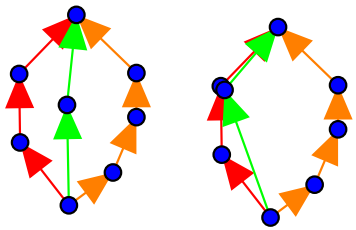
If there is only one element in $a \sqcup b$, then this element is $a \vee b$; it is the **infimum of all upper bounds of both** a **and** b;
If there is only one element in $a \sqcap b$, then this element is $a \wedge b$; it is the **supremum of all lower bounds of both** a **and** b.

Structure of (finite) lattices

The simplest examples of (finite) lattices are unions of directed paths P_{n_i} connecting the **bottom vertex** 0 and **top vertex** 1 with no other common vertices; we denote them $L = P_{n_1,n_2,\ldots,n_m}$.

Example

In figure left we have $P_{4,3,5}$.

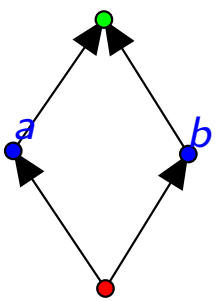


All other lattices may be obtained from them by identifying some of the vertices and edges of these maximal paths. The number m of them we call the **cut number** $c(\Gamma)$; the maximal length n_i of these walks is the **height** $h(\Gamma)$. It is easy to see:

$c(\Gamma) \times h(\Gamma) \geq$ cardinality of $V(\Gamma)$.

Example

The simplest nontrivial lattice, a **diamond**
$D(a, b) = L(0 \leq a, b \leq 1) = \{a, b\}_0^1 = P_{3,3}$ has cut number $c = 2$ and height $h = 3$. Here $a \vee b = 1$ and $a \wedge b = 0$. Replacing each directed edge with a copy of D we get its "powers"
$D^2, D^3, \ldots, D^n$ which are all lattices, too, and with cut number 2^n.



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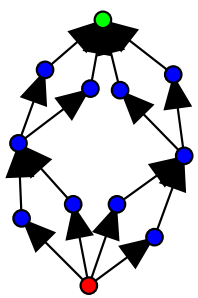

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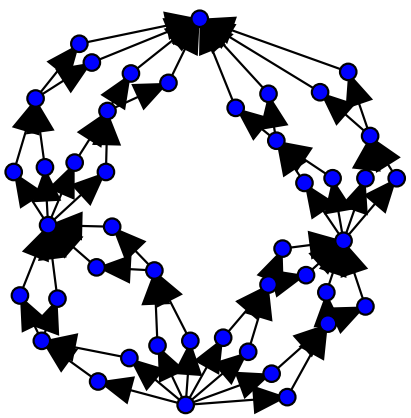
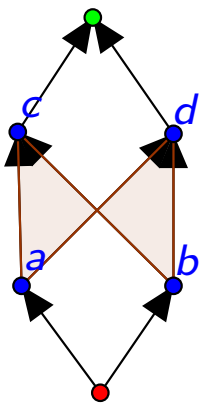


DDD

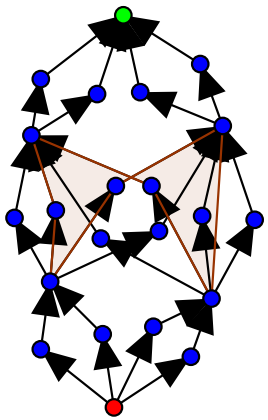
Example

The simplest nontrivial noncommutative lattice $N = B(a, b, c, d)_0^1$ consists of a **butterfly** $B(a, b, c, d) = \{a, b \leq c, d\}$ and additional top and bottom vertices 1 and 0. It has cut number 4 and height 4. It consists of 4 paths: (0,a,c,1), (0,a,d,1), (0,b,c,1), (0,b,d,1).

Here we can choose which of the points c or d will be $a \vee b$ and which $b \vee a$; likewise, we can choose, which of the points a or b will be $c \wedge b$ and which $b \wedge a$.



N



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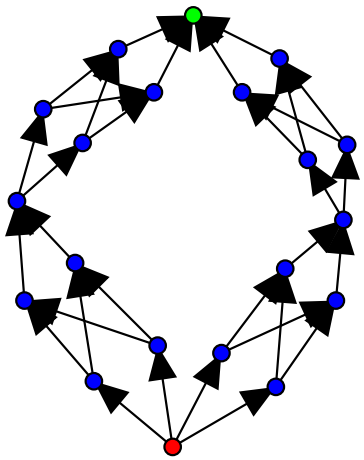

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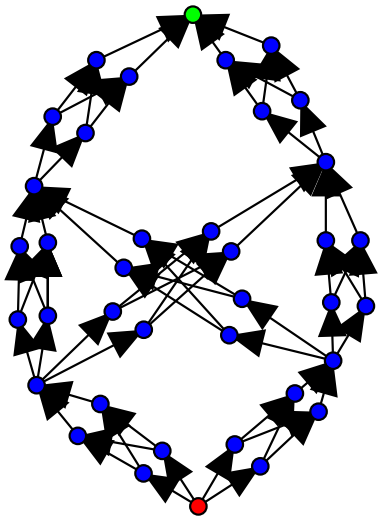


NN

Voltages on lattices

In order to get the idea for the "right" definition of a "covering lattice" let us first consider the following question:

Question 1. What happens when we apply the voltage construction to a directed graph $\Gamma = \Gamma(L)$, representing a lattice L?

$$L \to \Gamma(L) \to \Gamma^{\alpha}(L)$$

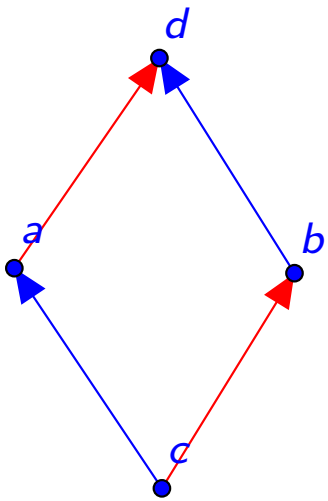
Does the derived graph $\Gamma^{\alpha}(L)$ always represent a lattice?

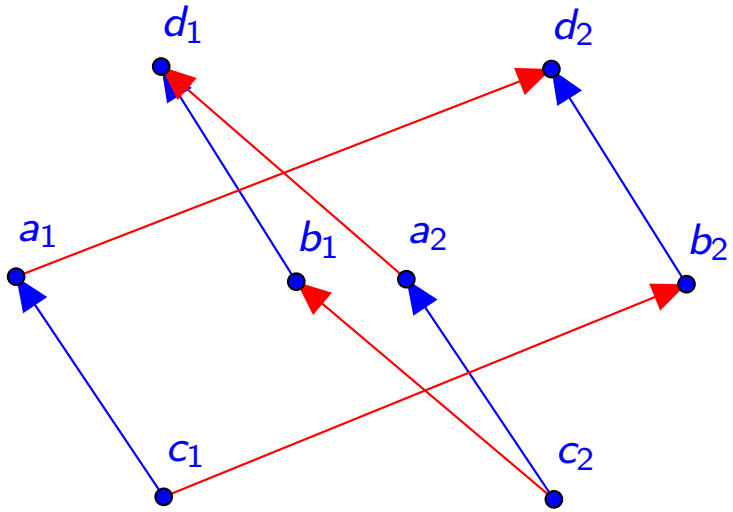
Let us investigate this!

Example

Let our base graph be the diamond $\Gamma = D(c \leq a, b \leq d)$
with voltages $\alpha \in \mathbb{G} = Z_2 = \{0, 1\}$ assigned to its arcs as follows:
voltage $\alpha = 0$ to the blue arcs $c \to a$ and $b \to d$
voltage $\alpha = 1$ to the red arcs $c \to b$ and $a \to d$.
The derived graph $\Gamma^\alpha = D(c_1 \leq a_1, b_2 \leq d_2) \cup D(c_2 \leq a_2, b_1 \leq d_1)$
does not represent a lattice (there is no $d_1 \vee d_2$; likewise, there is no $y \leq c_1, c_2$, hence there is no $c_1 \wedge c_2$).

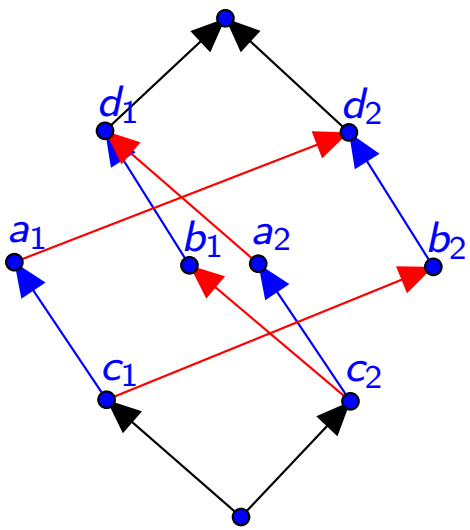
So we have a problem! *How to solve it?*



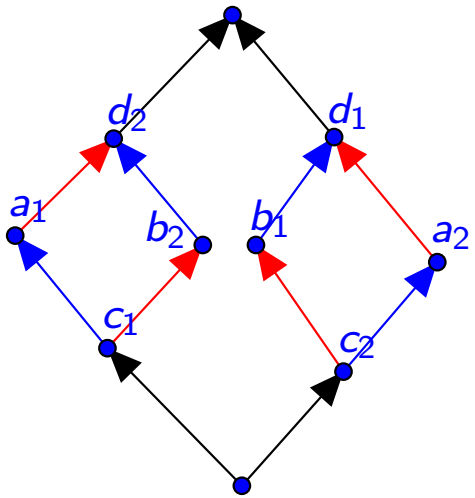


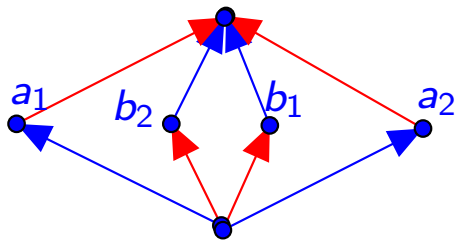
Two options for how to get a lattice from the Γ^α

1. Add 0 and 1 to get $(\Gamma^\alpha)_0^1$
2. Identify all top and all bottom points to get $(\Gamma^\alpha)^*$



$(\Gamma^\alpha)_0^1$





$(\Gamma^\alpha)^*$

Do our constructions always produce lattices from lattices?

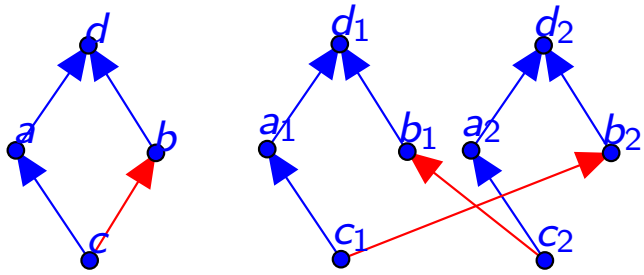
Question 2. If L is a lattice, do the directed graphs $(\Gamma(L)^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ always represent lattices, too?

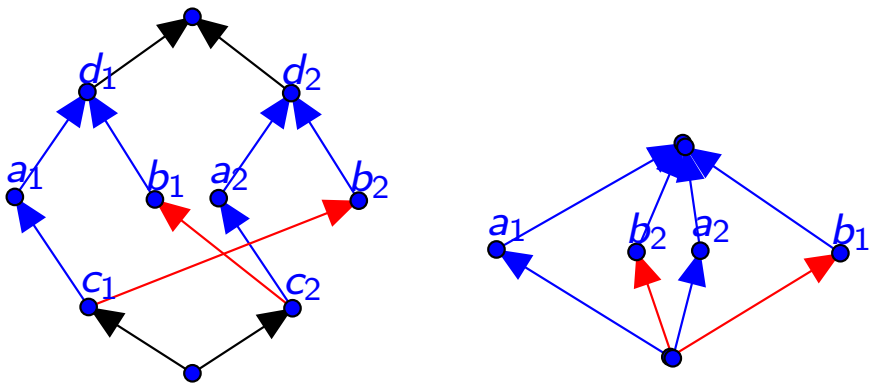
Let us investigate this!

Net voltage $1 \in \mathbb{Z}_2$ on the directed cycle

Now let the only nonzero voltage be $\alpha(c, b) = 1 \in \mathbb{Z}_2$:

Γ Γ^α

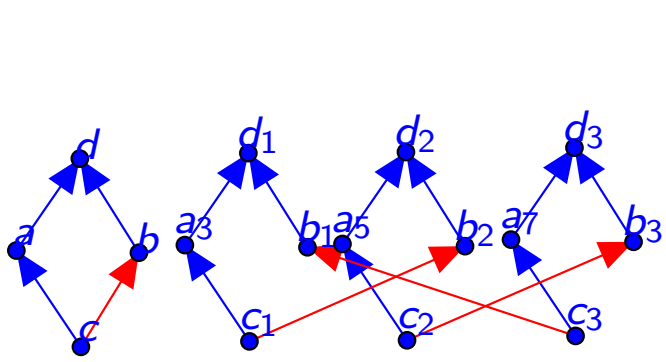




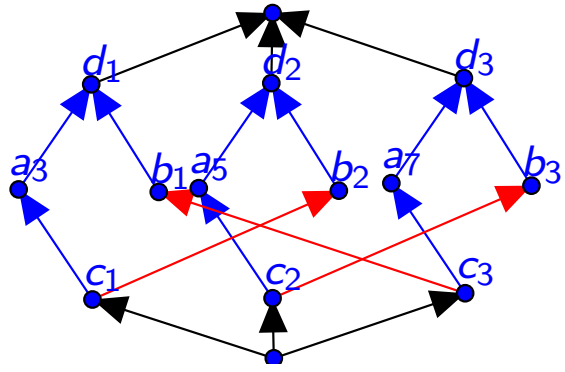
$(\Gamma^\alpha)_0^1$ $(\Gamma^\alpha)^*$

Now $(\Gamma^\alpha)_0^1$ is a skew lattice, and $(\Gamma^\alpha)^*$ is a lattice.

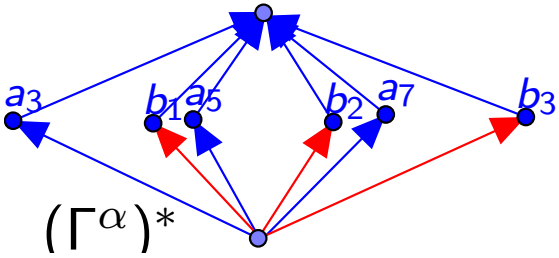
Voltages in $\mathbb{Z}_3$



Γ Γ^α



$(\Gamma^\alpha)_0^1$



$(\Gamma^\alpha)^*$

Here both $(\Gamma^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ are lattices. The same conclusion is true if we take the voltage $1 \in Z_n$ for any $n \geq 3$. However, if we take the voltage $1 \in Z$, then the construction of $(\Gamma^\alpha)^*$ is not possible at all (since there are no maximal and no minimal vertices in Γ^α).

Which lattices "lift" to lattices?

Question 3. Characterise those lattice graphs Γ which "lift" to Γ^α such that both $(\Gamma^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ are lattices.

To do this we need to refer to the concept of the **net voltage** [1] of a path or a cycle in Γ: it is just **the product of the voltages along the path or a cycle**. If the voltage group is commutative, then the net voltage of the cycle is independent of the choice of the initial vertex; otherwise, we get for the net voltage of a given cycle various conjugate elements in G, but the important thing here is that they all have the same **order** in the voltage group G.

Gross, Tucker 1987 [1]

If the order of the net voltage of the directed cycle C with length m in the base graph Γ is k, then it lifts to C^α which is the union of $(\mid V(\Gamma) \mid \times |G|)/k$ cycles of the length km.

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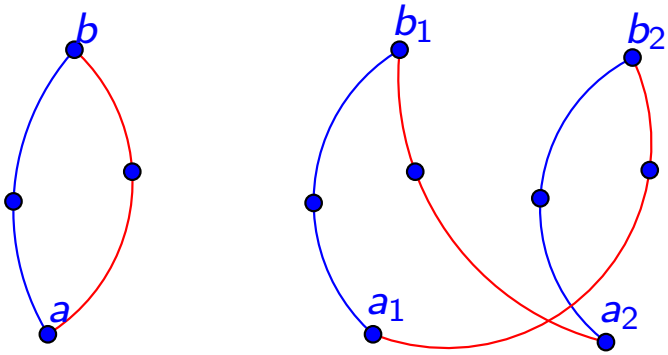
Let the base graph $\Gamma = \Gamma(L)$ represent a lattice L and let the voltages on its directed edges be taken from the group G.

Let the order $k = k(C)$ of the net voltage $\alpha(C) \in G$ of any of the directed cycles C in Γ be finite: $(\alpha(C))^k = id \in G$.

If $k(C) \neq 2$ for any directed cycle $C = a \nearrow b \searrow a$ in Γ, then the digraphs $(\Gamma^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ represent lattices, too;



Sketch of the proof.

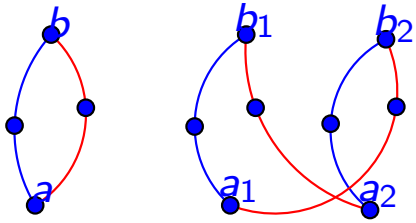
Step 1. It is enough to show that Γ^α contains no butterflies $B = a \nearrow c \searrow b \nearrow d \searrow a$ where a and b are not comparable and c and d are not comparable and there is no $e \in \{a, b\}^\uparrow \cap \{c, d\}^\downarrow$.

This is so because:

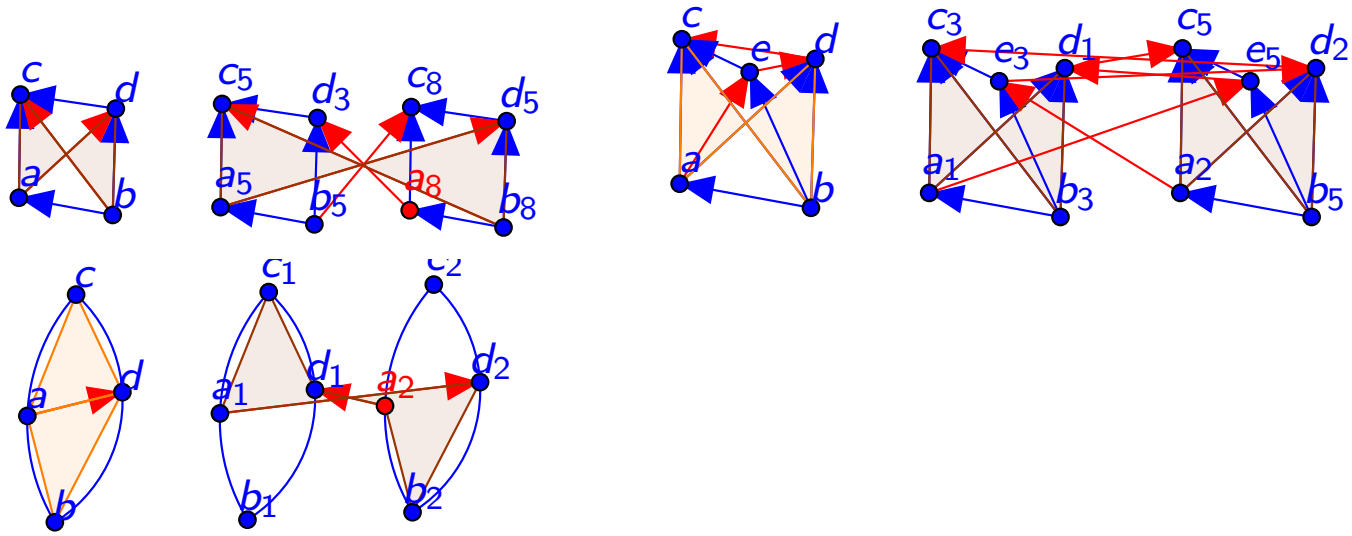
- Butterflies are characteristic for noncommutative lattices, hence they are forbidden in lattices.
- If there are no such B in Γ^α, then for any $a, b \in \Gamma^\alpha$ there is only one minimal vertex $x \in V(\Gamma^\alpha)$ such that $x \geq a, x \geq b$ and this is then $a \vee b$, and there is only one maximal vertex $y \in V(\Gamma^\alpha)$ such that $y \leq a, y \leq b$ and this is then $a \wedge b$.

Step 2. The forbidden butterflies may arise in Γ^α only from:

- the directed cycles $C = a \nearrow b \searrow a$ in Γ with net voltage of order 2 in the group G



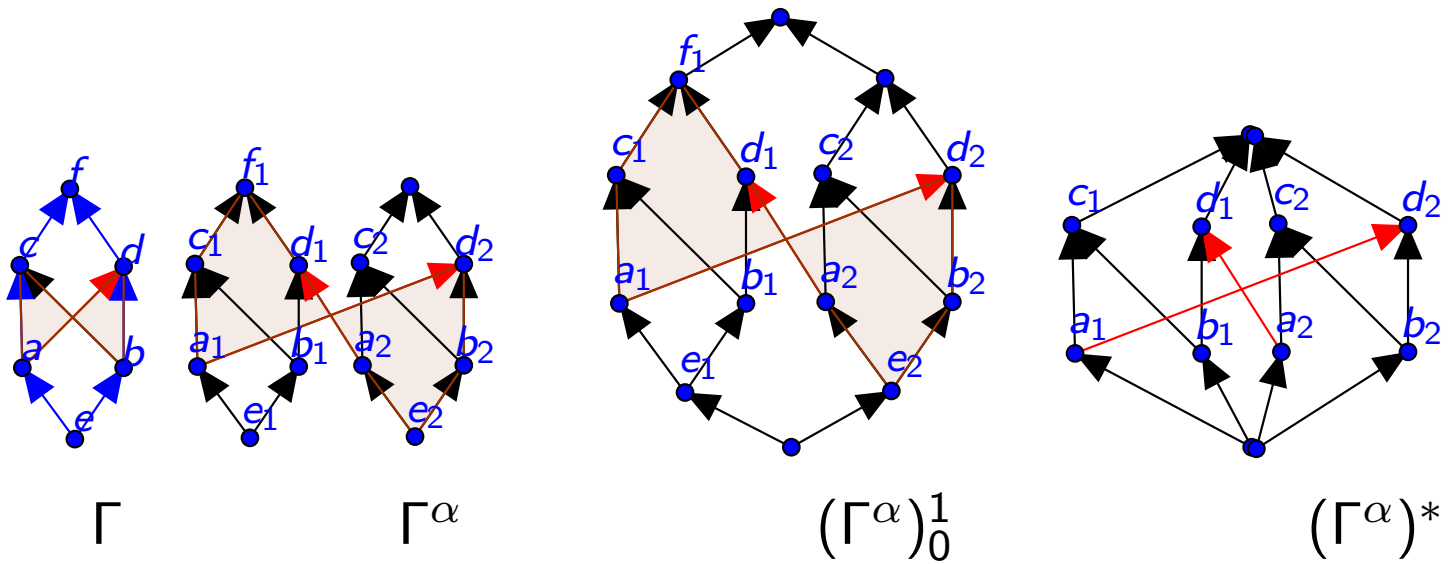
- or from various forms of "almost butterflies" $a \nearrow c \searrow b \nearrow d \searrow a$ with net voltage of order 1 such as



Step 3. But in all these cases we can find as a substructure a cycle $x \nearrow y \searrow x$ with the net voltage 1 of order 2, which is explicitly forbidden in the premises of our theorem.

Both skew lattices and lattices may be derived from a base skew lattice graph

Example



Γ Γ^{α} $(\Gamma^{\alpha})_0^1$ $(\Gamma^{\alpha})^*$

Here we have: skew lattice base graph $\Gamma(L)$ with butterfly $B(a \nearrow d \searrow b \nearrow c \searrow a)$

derived graph Γ^{α} and skew lattice graph $(\Gamma^{\alpha})_0^1$ with butterfly $B(a_1 \nearrow d_2 \searrow b_2 \searrow e_2 \nearrow a_2 \nearrow d_1 \nearrow f_1 \searrow c_1 \searrow a_1)$

lattice graph $(\Gamma^{\alpha})^*$

Comparison

K., 2021

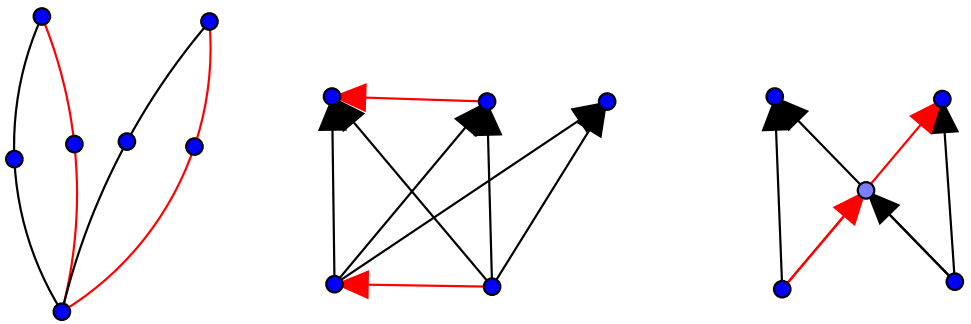
If $(\Gamma^\alpha)_0^1$ is a lattice graph,
then $(\Gamma^\alpha)^*$ is a lattice graph, too.

The opposite conclusion is not valid:
$(\Gamma^\alpha)^*$ may be a lattice graph and $(\Gamma^\alpha)_0^1$ not (see Example 4).

Proof. There are no butterflies in $(\Gamma^\alpha)_0^1$ hence also no butterflies in Γ^α. Can we obtain a butterfly by identifying the maximal vertices in Γ^α?
No vertex of a buterfly may be equal to the maximal vertex of $(\Gamma^\alpha)^*$.
But "below" this vertex everything in $(\Gamma^\alpha)^*$ is the same as in Γ^α where there are no butterflies.

When do our voltage construction produce skew lattices?

The idea of the theorem and the proof is the same as for lattices: It is enough that in Γ^α there are forbidden butterflies and that there are no $\{a, b\}$ such that $a \sqcup b$ or $a \sqcap b$ contains more than two elements. These structure may be "lifted" from Γ only from a few forbidden voltage structures. It turns out that there are essentially only three such forbidden structures, such as:

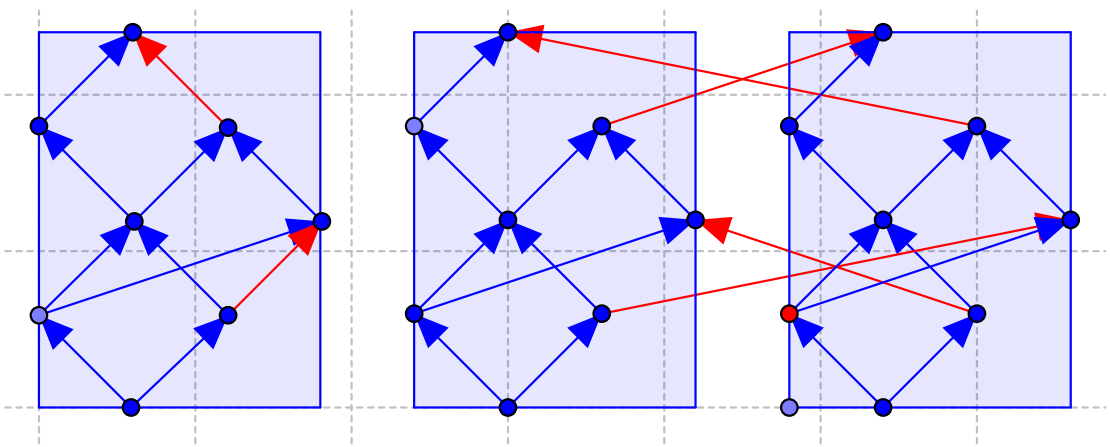


In the second and third case, there are some simple technical conditions on voltages that have to be applied on these structures if we want to obtain forbidden structure in Γ^α.

Parameters or properties of lattices preserved by voltage constructions

In transition from the lattice graph $\Gamma = \Gamma(L)$ to the derived graph Γ^α many parameters of the base lattice L are preserved, such as:

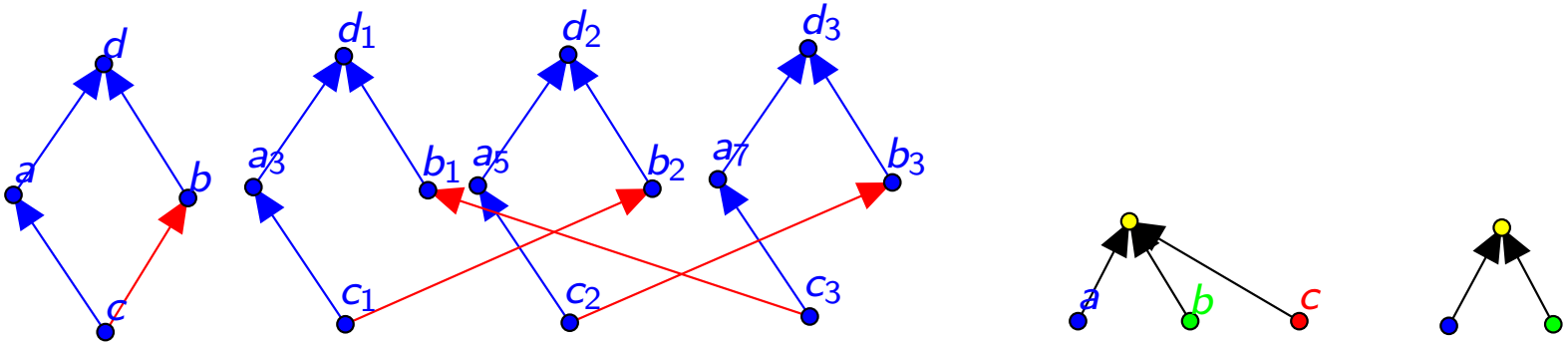
- the height $h(\Gamma) = h(\Gamma^\alpha)$ – the longest path
- the ratio $\frac{c(\Gamma)}{|V(\Gamma)|} = \frac{c(\Gamma^\alpha)}{|V(\Gamma^\alpha)|}$ between the cut number and the number of vertices
- the density $d = \frac{c(\Gamma)\times h(\Gamma)}{|V(\Gamma)|} = \frac{c(\Gamma^\alpha)\times h(\Gamma^\alpha)}{|V(\Gamma^\alpha)|}$



Here $h(\Gamma) = 5, c(\Gamma) = 7, \mid V(\Gamma) \mid = 8$ and $d = \frac{5\times 7}{8}$.

Under the conditions of Theorem 1 the following properties of the base lattice are preserved, too:

- the ratio $\frac{g(L)}{|V(L)|} = \frac{g(\Gamma^\alpha(L))}{|V(\Gamma^\alpha(L))|}$ where g is the number of **generators** – such vertices that all the other vertices may be expressed with them as their join $a \wedge b$ or meet $a \vee b$.



Here the generators of Γ are the vertices a and b, and the generators of Γ^α are their "lifts" $a_1, b_1, a_2, b_2, a_3, b_3$.

- the **cancellation property**: if $a \vee b = a \vee c$ then $b = c$ also "lifts" from $\Gamma(L)$ to $\Gamma^\alpha(L)$.

Symmetries and automorphisms of $(\Gamma^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ represented as geometrical structures in some $\mathbb{R}^n$

The existence theorems (e.g. Theorem 1 give necessary and sufficient conditions for $(\Gamma^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ to be a lattice (or a noncommutative lattice) in terms of forbidden minors in Γ.

Our constructions $(\Gamma^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ often produce graphs

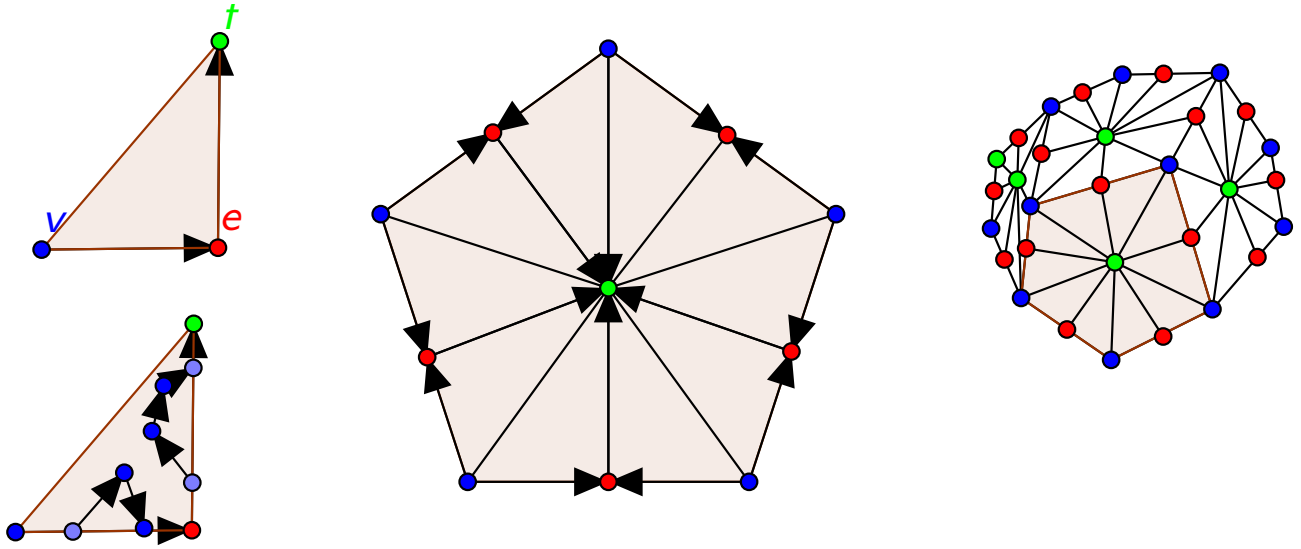
- which may be represented as geometric structures $\mathcal{L}$ (where vertices = points, edges = line segments) embedded in $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$ with various symmetries (i.e. isometries of $\mathbb{R}^n$ preserving $\mathcal{L}$); the corresponding symmetry group $Sym(\mathcal{L})$ depends on the dimension n of the ambiental space: $Sym_2(\mathcal{L}) \trianglelefteq \mathcal{S}\dagger\Updownarrow_\ni(\mathcal{L}) \cdots \trianglelefteq \mathcal{S}\dagger\Updownarrow_\setminus(\mathcal{L}) \ldots$.
- with many automorphisms (either "lifted" from the base graph Γ or "new"); in general, the automorphism group may be bigger than the symmetry group $Aut(\mathcal{L}) \trianglerighteq \mathcal{S}\dagger\Updownarrow(\mathcal{L})$.

Platonic lattices

Using other lifting techniques, e.g.

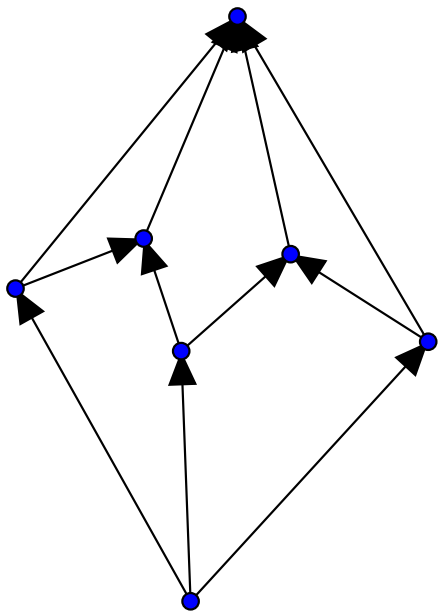
- using fundamental domains and symmetry groups of various surfaces,
- flag graphs of polyhedra nad maps with inscribed copies of the same directed graph

or using various iterative constructions with polyhedra etc. it is possible to obtain lattices with the symmetry of any Platonic solid (or any other polyhedron, or any surface, too):



Polyhedral lattices

From each polyhedron we can get a lattice by choosing its "south" (0) or "north" (1) pole, and directing all edges "rise" from 0 to 1.

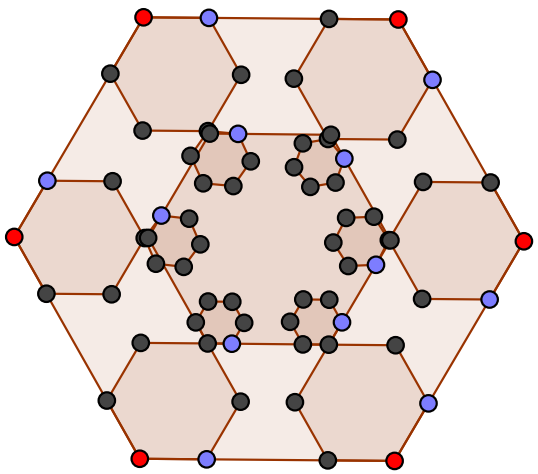


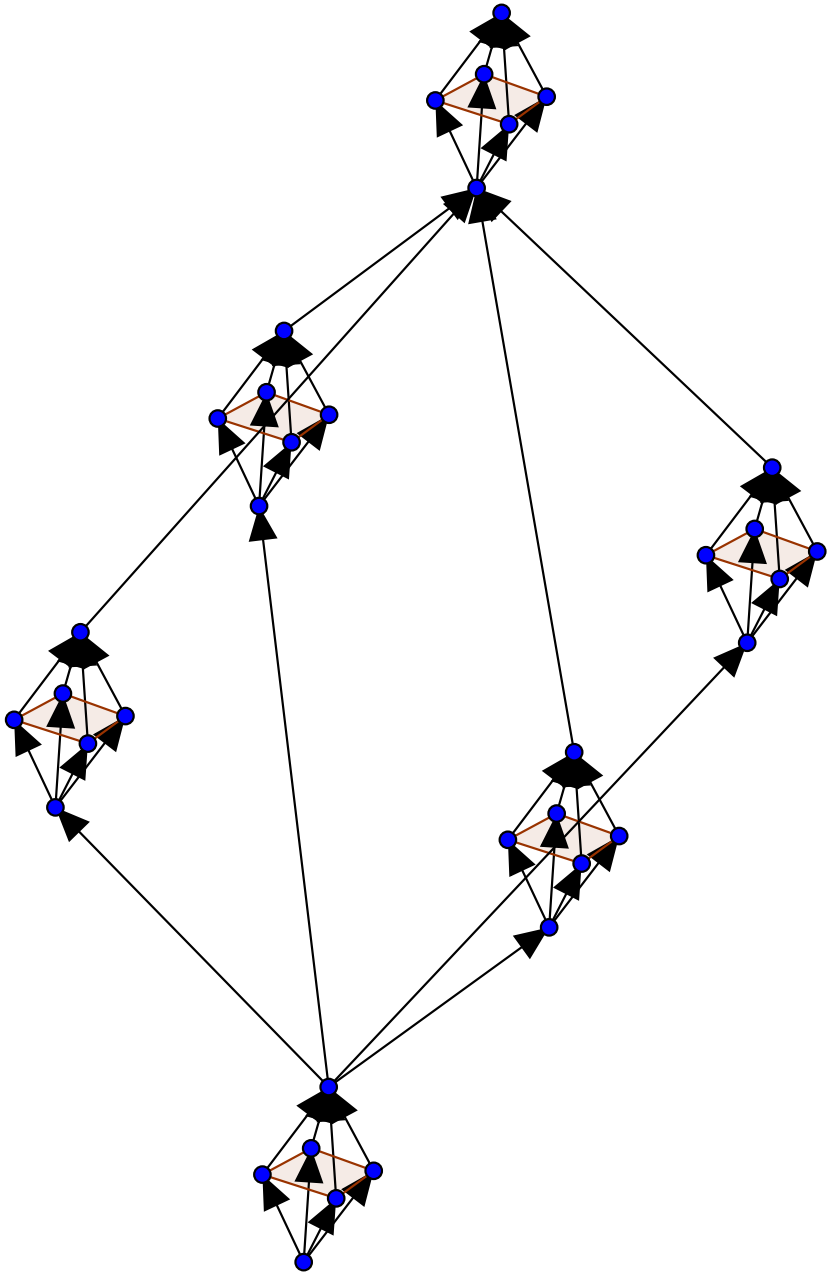
Iteratively generated lattices

Polygons and polyhedra allow various iterative constructions of lattices.

Example

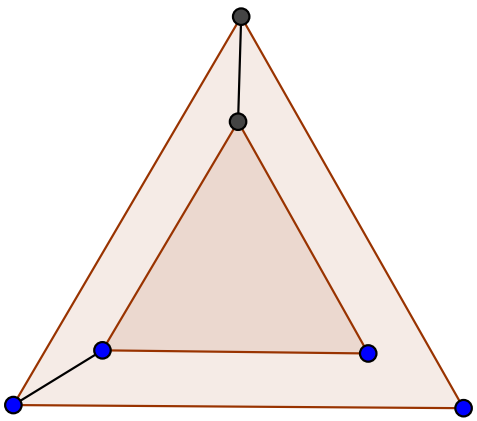
Place 6 copies of the "heagonal" or "octahedral" lattice in each of its vertices and connect them with the seventh copy touching all the previous six. Continue to get a "fractal" lattice "planar" or "spatial" lattice.

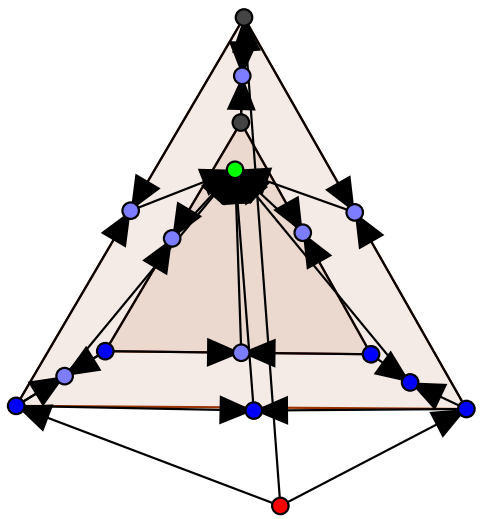




Lattices and automorphisms

Using Cayley graphs Γ of any given group G it is possible to get lattices L such that $Aut(L) = G$.
We just put on every edge of Γ two copies of the same lattice L and then add minimal and maximal vertices 0 and 1.
In fact, this works for any graph Γ.





Problem and result

I started this research after the first conference on noncommutative structures in Portorož (2019) with a question, presented in [3]:

- **How to define "voltage constructions on lattices" (to obtain "covering lattices" or "covering skew lattices")?**
- The original voltage construction, defined on (directed) graphs, fails for lattices, because the derived graph Γ^α of the base voltage graph Γ is never a lattice (since there are multiple maximal and minimal vertices); therefore, some adaptations of the original construction had to be made.

In this talk I have explained

- how the technique of voltage graphs naturally leads to **two different covering constructions** $(\Gamma^\alpha)_0^1$ and $(\Gamma^\alpha)^*$ **for lattices**.
- **when** (under which necessary and sufficient conditions) **the base (voltage) graph** $\Gamma = \Gamma(L)$ (representing either a lattice or a skew lattice) **"lifts" to a lattice** (or to a skew lattice).

Covering and voltage graphs

Some references:

- The concept of covering graphs and voltage graphs is in detail explained in [5], pp. 96–97.
- They were first introduced in [1] for *directed* graphs, but they may be applied also to *undirected* graphs, as in [6].
- Lifting graph automorphisms from the base graph is explained in [4].

So we have seen that lattices are interesting not only as algebraic objects but also as geometric objects.

Thank you!

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