Complex vs convex Morse functions and geodesic open books

Pierre Dehornoy & Burak Özbağcı

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Σ : closed oriented surface, with a **Riemannian metric**

$(T^*\Sigma, \omega)$: cotangent bundle with its **canonical symplectic structure** $\omega = d\lambda$, (where λ is the **Liouville** one form)

$DT^*\Sigma$: disk cotangent bundle and $(ST^*\Sigma, \xi)$: unit cotangent bundle with its **canonical contact structure** $\xi = \ker \lambda$

In the literature, there are ***four distinct*** constructions of **open books** adapted to $(ST^*\Sigma, \xi)$ with the following flavors :

- **dynamical** (Birkhoff 1917, Fried 1983)
- **complex** (A'Campo 1999, Ishikawa 2004)
- **contact** (Giroux 1991, 2002, Massot 2012)
- **symplectic** (Seidel 2000, Johns 2012)

Theorem [Dehornoy - O. , 2021] :

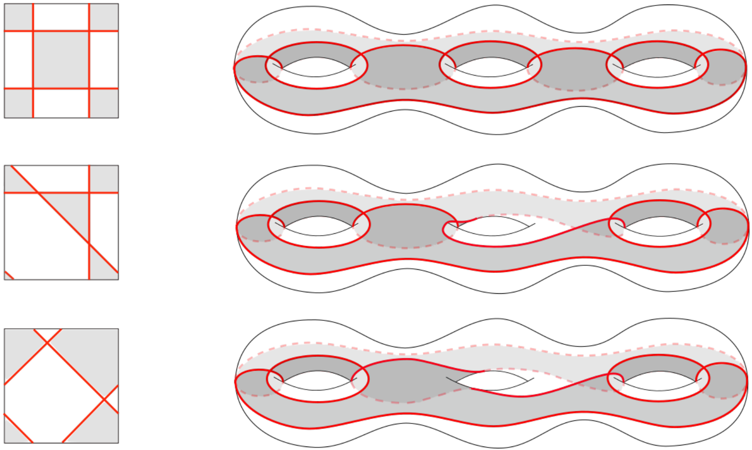
For a fixed ***initial data*** on the surface Σ, all **open books** for the unit cotangent bundle $ST^*\Sigma$ obtained by these **four constructions** are pairwise **isomorphic**.

Initial data is either an **admissible divide** or an **ordered Morse function** on Σ.

Admissible divides and adapted Morse functions

A **divide** $P \subset \Sigma$ is a generic immersion of the disjoint union of finitely many copies of the unit **circle.** P is called **admissible** if

- P is connected,
- each component of $\Sigma \setminus P$ is simply connected and
- $\Sigma \setminus P$ admits a **black-and-white** coloring.



Three different ***admissible divides*** depicted for a **torus** (left) and on a **genus** 3 **surface** (right).

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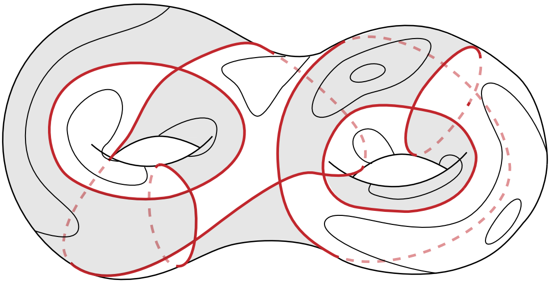
An **ordered Morse function** $f : \Sigma \to \mathbb{R}$ is **adapted** to an admissible divide $P \subset \Sigma$ if

- $P = f^{-1}(0)$,
- each **double point** of P corresponds to a **critical point** of f of **index 1**, and
- each **black** (resp. white) region of $\Sigma \setminus P$ contains exactly one **index** 2 (resp. 0) critical point of f.



An ***admissible divide*** P on a genus 2 surface,
a **black-and-white** coloring of $\Sigma \setminus P$, and
some level sets of an ordered Morse function **adapted** *to* P.

Four distinct constructions of open books 1917—2012

Construction 1

A'Campo-Ishikawa open book :

Complexification of a Morse function

Ishikawa (2004) constructed **Lefschetz fibrations** $DT^*\Sigma \to D^2$

based on **A'Campo**'s work.

Each Lefschetz fibration $DT^*\Sigma \to D^2$ induces an **open book**

on the boundary $\partial(DT^*\Sigma) = ST^*\Sigma$.

Complexification of the Morse function adapted to P

For any admissible divide $P \subset \Sigma$, there is an **ordered Morse** function $f_P : \Sigma \to \mathbb{R}$ **adapted** to P, which in turn, gives an "***almost complexified Morse function***" $F_P : T^*\Sigma \to \mathbb{C}$ defined as

$$F_P(x,u) = f_P(x) + i df_P(x)(u) - \frac{1}{2}\chi(x) H_{f_P}(x)(u,u)$$

such that a point in $T^*\Sigma$ is a critical point of F_P if and only if it belongs to Σ and it is a critical point of f_P.

$F_P : T^*\Sigma \to \mathbb{C}$ descents to a **Lefschetz fibration**

$$\pi_P : DT^*\Sigma \to D^2$$

Ishikawa's Lefschetz fibrations $\pi_P : DT^*\Sigma \to D^2$

The **singular fiber** over $0 \in D^2$ contains a **singularity** for each **double point** of P.

Corresponding to each **index 2** (resp. 0) critical point of f_p, there is a singular fiber of π_P containing a **unique** singularity.

Moreover, the **regular fiber** and the **monodromy** of the Lefschetz fibration

$$\pi_P : DT^*\Sigma \to D^2$$

can be given by a method due to **A'Campo '99**.

Construction 2

A symplectic point of view due to Johns

Johns (2012) constructed

exact symplectic Lefschetz fibrations $(DT^*\Sigma, \omega) \to D^2$

based on the work of **Seidel**.

Each Lefschetz fibration $DT^*\Sigma \to D^2$ induces an **open book**

on the boundary $\partial(DT^*\Sigma) = ST^*\Sigma$, which **supports** ξ.

Johns' Lefschetz fibrations $DT^*\Sigma \to D^2$

Let $f : \Sigma \to \mathbb{R}$ be a Morse function.

Based on the **handle decomposition** of the surface Σ induced by f, **Johns** constructs an **exact symplectic** Lefschetz fibration $\pi : E \to D^2$ with explicit (Lagrangian) vanishing cycles satisfying the following :

- There is an **exact Lagrangian** embedding $\Sigma \subset E$
- **Critical** points of π are in Σ, $\pi|_\Sigma = f : \Sigma \to \mathbb{R}$, and
- E is ***conformally exact symplectomorphic*** to $(DT^*\Sigma, \omega = d\lambda)$.

The fiber and the vanishing cycles

The construction of the **fiber** depends on $f : \Sigma \to \mathbb{R}$ and for each **critical point** of f, there is a **vanishing cycle** of the fibration.

More precisely, the fiber is constructed by a **plumbing of some annuli**, each one of which represents a neighborhood of a vanishing cycle corresponding to a critical point of f of **index zero or one.**

Moreover, the vanishing cycles corresponding to the critical points of f of **index two** are obtained from the vanishing cycles corresponding to the critical points of f of index zero and one, by a simultaneous ***Lagrangian surgery*** where they meet on the fiber.

Construction 3

Giroux open book :

Convexity in contact topology

Giroux's theorem on the convexity of contact manifolds

ξ : **any** contact structure on a closed manifold V^{2n+1}

X : a **contact** vector field on V (i.e. whose flow preserves ξ), which is **gradient-like** for an **ordered** Morse function $F : V \to \mathbb{R}$

($F : V \to \mathbb{R}$ is called ξ-**convex**, and (V, ξ) is called **convex** contact manifold.)

L : a **regular level set** of F above the critical values of index n and below the critical values of index $n+1$.

C_X : **characteristic hypersurface** of X. (i.e set of points where X is **tangent** to ξ).

Then, L is **transverse** to C_X,

$K = L \cap C_X$ is the **binding** of an open book that **supports** ξ, and

K cuts $L \cup C_X$ into ***four pages*** of this open book.

Bundle of cooriented hyperplanes

V^{2n+1} : bundle of **cooriented hyperplanes** tangent to a smooth manifold M^{n+1}

ξ : the **canonical contact structure** on V

(V can be identified with ST^*M. Then ξ is given by the kernel of λ.)

Starting from a Morse function $f : M^{n+1} \to \mathbb{R}$, **Giroux** (1991) constructs a ξ-convex Morse function F on ST^*M, which proves that (ST^*M, ξ) is **convex.**

(*Giroux (2002) proved that* ***every closed contact manifold is convex !***)

Massot (2012) shows that, in dimension three ($n = 1$), if $f : \Sigma \to \mathbb{R}$ is an ordered Morse function, then ξ-convex Morse function F on $ST^*\Sigma$ is **ordered** and thus gives an **open book** adapted to $(ST^*\Sigma, \xi)$ by Giroux's theorem.

Construction 4

Birkhoff cross sections and

geodesic open books

Birkhoff cross sections

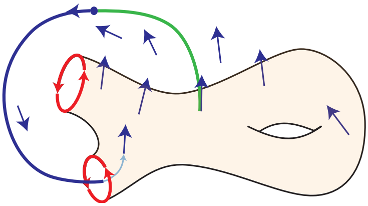
Suppose that X is a **nonsingular** vector field on a closed and oriented 3-manifold M. A ***Birkhoff cross section*** for (M, X) is a compact orientable surface S with boundary such that

- S is embedded in M,
- X is **transverse** to the interior of S,
- the boundary ∂S is **tangent** to X, and
- **every orbit** of X intersects S after a bounded time.

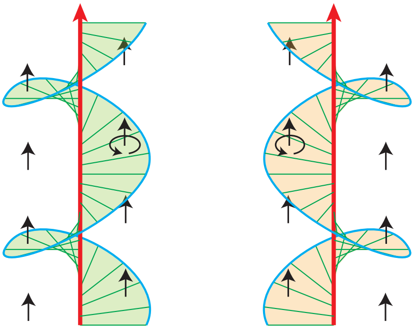
$\Longrightarrow$ ∂S is the union of finitely many **periodic orbits** of X.

A Birkhoff cross section S is said to be a ***positive*** (resp. ***negative***) if for each component of ∂S, the natural orientation given by X ***coincides with*** (resp. is ***opposite of***) its orientation inherited as the boundary of S.

A **positive** Birkhoff cross section S for (M, X) is a page of an **open book** for M whose oriented binding is ∂S.



Birkhoff cross section



How it looks around boundary components, for a **negative** (left) and a **positive** (right) Birkhoff cross section, respectively.

Geodesic flow and geodesic open books

The **geodesic flow** Φ is the flow on the unit ***tangent*** bundle $ST\Sigma$, whose **orbits** are the lifts of the **geodesics** on Σ. More precisely, if g is a geodesic with unit speed on the Riemann surface Σ, then the orbit of Φ going through $(g(0), \dot{g}(0))$ is given by

$$\Phi^t(g(0), \dot{g}(0)) = (g(t), \dot{g}(t)).$$

We say that an admissible divide P on a surface Σ is ***convex*** if every curve in P is a **closed geodesic**, every geodesic on Σ intersects P in bounded time, and every region of $\Sigma \setminus P$ can be foliated by concentric curves with non-vanishing curvature.

Theorem : (**Birkhoff** (1917) & **Fried** (1983))

If $P \subset \Sigma$ is ***convex***, then the lift of P to $ST\Sigma$ is the binding of an open book — called the ***geodesic open book***, whose pages are ***negative*** Birkhoff cross sections of the geodesic flow.

There is a corresponding **cogeodesic** flow on $ST^*\Sigma$, which coincides with the **Reeb** flow for λ.

The pages of the ***cogeodesic open book***, are **positive** Birkhoff cross sections of the cogeodesic flow.

$\Longrightarrow$ the ***cogeodesic open book*** **supports** $(ST^*\Sigma, \xi)$.

Theorem [Dehornoy - O. , 2021] :

For a fixed ***initial data*** on the surface Σ, all **open books** for the unit cotangent bundle $ST^*\Sigma$ obtained by these **four constructions** are pairwise **isomorphic**.

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Outline of proof :

- Ishikawa's Lefschetz fibration $\Longleftrightarrow$ Johns' Lefschetz fibration
- Giroux open book $\Longleftrightarrow$ A'Campo-Ishikawa open book
- A'Campo-Ishikawa open book $\Longleftrightarrow$ geodesic open book

Minimal genus open books

Theorem [Dehornoy - O. , 2021] : Up to homeomorphism, there are **exactly three admissible divides** on a closed and oriented surface Σ of genus at least one, as depicted below, that yield **genus one open books** obtained by any of the four methods. These open books have $4g, 4g+2$, and $4g+4$ binding components, respectively.