Complex vs convex Morse functions and geodesic open books

Pierre Dehornoy & Burak Özbağcı

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Σ : closed oriented surface, with a **Riemannian metric**

\((T^*Σ, \omega)\) : cotangent bundle with its **canonical symplectic structure**

\(\omega = d\lambda\), (where \(\lambda\) is the **Liouville** one form)

\(DT^*Σ\) : disk cotangent bundle and \((ST^*Σ, \xi)\) : unit cotangent bundle with its **canonical contact structure** \(\xi = \ker\lambda\)

In the literature, there are **four distinct** constructions of **open books** adapted to \((ST^*Σ, \xi)\) with the following flavors :

- **dynamical** (Birkhoff 1917, Fried 1983)
- **contact** (Giroux 1991, 2002, Massot 2012)
- **symplectic** (Seidel 2000, Johns 2012)
Theorem [Dehornoy - O. , 2021] :

For a fixed initial data on the surface $\Sigma$, all open books for the unit cotangent bundle $ST^*\Sigma$ obtained by these four constructions are pairwise isomorphic.

Initial data is either an admissible divide or an ordered Morse function on $\Sigma$. 
Admissible divides and adapted Morse functions

A divide $P \subset \Sigma$ is a generic immersion of the disjoint union of finitely many copies of the unit circle. $P$ is called admissible if

- $P$ is connected,
- each component of $\Sigma \setminus P$ is simply connected and
- $\Sigma \setminus P$ admits a black-and-white coloring.
Three different *admissible divides* depicted for a *torus* (left) and on a *genus 3 surface* (right).
A **divide** $P \subset \Sigma$ is a generic immersion of the disjoint union of finitely many copies of the unit **circle**. $P$ is called **admissible** if

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An **ordered Morse function** $f : \Sigma \to \mathbb{R}$ is **adapted** to an admissible divide $P \subset \Sigma$ if

- $P = f^{-1}(0)$,
- each **double point** of $P$ corresponds to a **critical point** of $f$ of **index** 1, and
- each **black** (resp. white) region of $\Sigma \setminus P$ contains exactly one **index** 2 (resp. 0) critical point of $f$. 
An *admissible divide* $P$ on a genus 2 surface, a *black-and-white* coloring of $\Sigma \setminus P$, and some level sets of an ordered Morse function *adapted* to $P$. 
Four distinct constructions of open books 1917—2012
Construction 1

A’Campo-Ishikawa open book:

Complexification of a Morse function

Ishikawa (2004) constructed Lefschetz fibrations $DT^*\Sigma \rightarrow D^2$

based on A’Campo’s work.

Each Lefschetz fibration $DT^*\Sigma \rightarrow D^2$ induces an open book

on the boundary $\partial(DT^*\Sigma) = ST^*\Sigma$. 
Complexification of the Morse function adapted to $P$

For any admissible divide $P \subset \Sigma$, there is an ordered Morse function $f_P : \Sigma \to \mathbb{R}$ adapted to $P$, which in turn, gives an “almost complexified Morse function” $F_P : T^*\Sigma \to \mathbb{C}$ defined as

$$F_P(x, u) = f_P(x) + idf_P(x)(u) - \frac{1}{2} \chi(x) H_{f_P}(x)(u, u)$$

such that a point in $T^*\Sigma$ is a critical point of $F_P$ if and only if it belongs to $\Sigma$ and it is a critical point of $f_P$.

$F_P : T^*\Sigma \to \mathbb{C}$ descents to a Lefschetz fibration

$$\pi_P : DT^*\Sigma \to D^2$$
Ishikawa’s Lefschetz fibrations $\pi_P : DT^*\Sigma \to D^2$

The **singular fiber** over $0 \in D^2$ contains a **singularity** for each **double point** of $P$.

Corresponding to each **index 2** (resp. 0) critical point of $f_P$, there is a singular fiber of $\pi_P$ containing a **unique** singularity.

Moreover, the **regular fiber** and the **monodromy** of the Lefschetz fibration

$$\pi_P : DT^*\Sigma \to D^2$$

can be given by a method due to **A’Campo ’99**.
Construction 2

A symplectic point of view due to Johns

Johns (2012) constructed exact symplectic Lefschetz fibrations \((DT^*\Sigma, \omega) \rightarrow D^2\) based on the work of Seidel.

Each Lefschetz fibration \(DT^*\Sigma \rightarrow D^2\) induces an open book on the boundary \(\partial(DT^*\Sigma) = ST^*\Sigma\), which supports \(\xi\).
Johns’ Lefschetz fibrations $DT^*\Sigma \to D^2$

Let $f : \Sigma \to \mathbb{R}$ be a Morse function.

Based on the handle decomposition of the surface $\Sigma$ induced by $f$, Johns constructs an exact symplectic Lefschetz fibration $\pi : E \to D^2$ with explicit (Lagrangian) vanishing cycles satisfying the following:

- There is an exact Lagrangian embedding $\Sigma \subset E$
- Critical points of $\pi$ are in $\Sigma$, $\pi|_{\Sigma} = f : \Sigma \to \mathbb{R}$, and
- $E$ is conformally exact symplectomorphic to $(DT^*\Sigma, \omega = d\lambda)$. 
The fiber and the vanishing cycles

The construction of the fiber depends on $f : \Sigma \to \mathbb{R}$ and for each critical point of $f$, there is a vanishing cycle of the fibration.

More precisely, the fiber is constructed by a plumbing of some annuli, each one of which represents a neighborhood of a vanishing cycle corresponding to a critical point of $f$ of index zero or one.

Moreover, the vanishing cycles corresponding to the critical points of $f$ of index two are obtained from the vanishing cycles corresponding to the critical points of $f$ of index zero and one, by a simultaneous Lagrangian surgery where they meet on the fiber.
Construction 3

Giroux open book:

*Convexity in contact topology*
Giroux’s theorem on the convexity of contact manifolds

$\xi$: any contact structure on a closed manifold $V^{2n+1}$

$X$: a contact vector field on $V$ (i.e. whose flow preserves $\xi$), which is gradient-like for an ordered Morse function $F : V \rightarrow \mathbb{R}$

($F : V \rightarrow \mathbb{R}$ is called $\xi$-convex, and $(V, \xi)$ is called convex contact manifold.)

$L$: a regular level set of $F$ above the critical values of index $n$ and below the critical values of index $n + 1$.

$C_X$: characteristic hypersurface of $X$. (i.e set of points where $X$ is tangent to $\xi$).

Then, $L$ is transverse to $C_X$,

$K = L \cap C_X$ is the binding of an open book that supports $\xi$, and $K$ cuts $L \cup C_X$ into four pages of this open book.
Bundle of cooriented hyperplanes

$V^{2n+1} :$ bundle of cooriented hyperplanes tangent to a smooth manifold $M^{n+1}$

$\xi :$ the canonical contact structure on $V$

($V$ can be identified with $ST^* M$. Then $\xi$ is given by the kernel of $\lambda$.)

Starting from a Morse function $f : M^{n+1} \to \mathbb{R}$, Giroux (1991) constructs a $\xi$-convex Morse function $F$ on $ST^* M$, which proves that $(ST^* M, \xi)$ is convex.

(Giroux (2002) proved that every closed contact manifold is convex!)

Massot (2012) shows that, in dimension three ($n = 1$), if $f : \Sigma \to \mathbb{R}$ is an ordered Morse function, then $\xi$-convex Morse function $F$ on $ST^* \Sigma$ is ordered and thus gives an open book adapted to $(ST^* \Sigma, \xi)$ by Giroux’s theorem.
Construction 4

Birkhoff cross sections and
geodesic open books
Birkhoff cross sections

Suppose that $X$ is a nonsingular vector field on a closed and oriented 3-manifold $M$. A Birkhoff cross section for $(M, X)$ is a compact orientable surface $S$ with boundary such that

- $S$ is embedded in $M$,
- $X$ is transverse to the interior of $S$,
- the boundary $\partial S$ is tangent to $X$, and
- every orbit of $X$ intersects $S$ after a bounded time.

$\implies \partial S$ is the union of finitely many periodic orbits of $X$.

A Birkhoff cross section $S$ is said to be a positive (resp. negative) if for each component of $\partial S$, the natural orientation given by $X$ coincides with (resp. is opposite of) its orientation inherited as the boundary of $S$.

A positive Birkhoff cross section $S$ for $(M, X)$ is a page of an open book for $M$ whose oriented binding is $\partial S$. 
Birkhoff cross section

How it looks around boundary components, for a **negative** (left) and a **positive** (right) Birkhoff cross section, respectively.
Geodesic flow and geodesic open books

The **geodesic flow** $\Phi$ is the flow on the unit **tangent** bundle $ST\Sigma$, whose **orbits** are the lifts of the **geodesics** on $\Sigma$. More precisely, if $g$ is a geodesic with unit speed on the Riemann surface $\Sigma$, then the orbit of $\Phi$ going through $(g(0), \dot{g}(0))$ is given by

$$\Phi^t(g(0), \dot{g}(0)) = (g(t), \dot{g}(t)).$$

We say that an admissible divide $P$ on a surface $\Sigma$ is **convex** if every curve in $P$ is a **closed geodesic**, every geodesic on $\Sigma$ intersects $P$ in bounded time, and every region of $\Sigma \setminus P$ can be **foliated** by concentric curves with non-vanishing curvature.

**Theorem :** *(Birkhoff (1917) & Fried (1983))*

If $P \subset \Sigma$ is **convex**, then the lift of $P$ to $ST\Sigma$ is the binding of an open book — called the **geodesic open book**, whose pages are **negative** Birkhoff cross sections of the geodesic flow.
There is a corresponding cogeodesic flow on $ST^*\Sigma$, which coincides with the Reeb flow for $\lambda$.

The pages of the cogeodesic open book, are positive Birkhoff cross sections of the cogeodesic flow.

$\implies$ the cogeodesic open book supports $(ST^*\Sigma, \xi)$. 
**Theorem** [Dehornoy - O., 2021]:

For a fixed *initial data* on the surface $\Sigma$, all *open books* for the unit cotangent bundle $ST^*\Sigma$ obtained by these *four constructions* are pairwise *isomorphic*.

**Initial data** is either an *admissible divide* or an *adapted Morse function* on $\Sigma$.

**Outline of proof**:

- Ishikawa’s Lefschetz fibration $\iff$ Johns’ Lefschetz fibration
- Giroux open book $\iff$ A’Campo-Ishikawa open book
- A’Campo-Ishikawa open book $\iff$ geodesic open book
**Minimal genus open books**

**Theorem** [Dehornoy - O., 2021]: Up to homeomorphism, there are exactly three admissible divides on a closed and oriented surface $\Sigma$ of genus at least one, as depicted below, that yield genus one open books obtained by any of the four methods. These open books have $4g$, $4g + 2$, and $4g + 4$ binding components, respectively.