Nonstandard finite elements for wave problems

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**PDE problem - variational formulation**

\[ \mathcal{L} u = f \quad \text{in } \Omega \quad \text{with boundary conditions} \]

find \( u \in V : \ a(u, v) = \ell(v) \quad \forall \ v \in V \)

**Galerkin method**

find \( u_N \in V_N \subset V : \ a(u_N, v_N) = \ell(v_N) \quad \forall \ v_N \in V_N, \quad \text{dim}(V_N) = N < +\infty \)

\[ Au = b \quad N \times N \text{ algebraic linear system} \]

- \( V_N \subset V \) (conformity)
- same \( a(\cdot, \cdot) \) and \( \ell(\cdot) \) as for the continuous problem
Standard finite elements

choice of $V_N$

- partition $\mathcal{T}_h$ of $\Omega$ into simplices or bricks
- $V_p(\mathcal{T}_h) =$ space of piecewise polynomials of degree $p$ on $\mathcal{T}_h$ such that $V_p(\mathcal{T}_h) \subset V$ (conformity)

$p = 1$ \hspace{1cm} $p = 2$ \hspace{1cm} $p = 3$

time-dependent problems

- finite elements in space $\rightarrow$ system of ODEs \hspace{1cm} e.g. $\dot{u} + Au(t) = b(t)$
- ODE solver (time-stepping)
highly oscillatory solutions $\rightarrow$ high resolution is required (high $p$ in space convenient)
multimaterials $\rightarrow$ different resolutions in different subdomains
solution singularities $\rightarrow$ locally refined spatial meshes required
the presence of small spatial elements forces small time-steps to avoid instabilities
- CFL condition

nonstandard / ad hoc approaches?
**standard**

- variational formulation in space and time-stepping
- mesh in space \( \times \) partition of the time interval
- Galerkin setting: \( V_N \subset V, \quad a_N(\cdot, \cdot) = a(\cdot, \cdot), \quad \ell_N(\cdot) = \ell(\cdot) \)
- local basis functions in complete polynomial spaces

**nonstandard**

- space-time variational formulation
- subcharacteristic, tent-shaped meshes of the space-time domain
- discontinuous Galerkin setting: \( V_N \not\subset V, \quad a_N(\cdot, \cdot) \neq a(\cdot, \cdot), \quad \ell_N(\cdot) \neq \ell(\cdot) \)
- local basis functions chosen depending on the PDE operator
why space-time?

- *high-order* approximation in both space and time is simple to obtain
- stability is achieved under a *local* CFL condition
- the numerical solution is available at *all times* in \((0, T)\)

**drawback:** high complexity

- time dependent problem in \(d\) space dimensions \(\rightarrow (d + 1)\)-dimensional problem
model problem: the acoustic wave equation

space-time discontinuous Galerkin (DG) discretization

reduction of the complexity:

- special (Trefftz) basis functions and tent-pitched meshes [1], [2]
- tensor-product (in time) elements and combination formula [3]


Some references

space-time finite element methods for wave problems

- early works (FEM): [Hughes, Hulbert, 1988, 1990], [French, 1993], [Johnson, 1993], ...

- DG: [Falk, Richter, 1999], [Yin, Acharya, Sobh, Haber, Tortorelli, 2000],
  [Monk, Richter, 2005], [Costanzo, Huang, 2005], [Abedi, Petracovici, Haber, 2006],
  [van der Vegt, 2006], [Feistauer, Hájek, Švadlenka, 2007], ...
  [Gopalakrishnan, Monk, Sepúlveda, 2015], [Dörfler, Findeisen, Wieners, 2016],
  [Gopalakrishnan, Schöberl, Wintersteiger, 2017, 2019], ...

- Trefftz: [Maciąg, Wauer, Sokala, 2005–2011], [Liu, Kuo, 2016],
  [Petersen, Farhat, Tezaur, 2009], [Wang, Tezaur, Farhat, 2014]
  [Egger, Kretzschmar, Schnep, Tzukermann, Weiland, 2014, 2015],
  [Banjai, Georgoulis, Lijoka, 2017], [Barucq, Calandra, Diaz, Shishenina, 2018, 2020],
  [1], [2]

- recent, on tensor-product meshes: [Steinbach, Zank, 2019], [Ernesti, Wieners, 2019], [3]
Model problem

the acoustic wave problem as a 1st order system

\[ Q = \Omega \times (0, T), \quad \Omega \subset \mathbb{R}^d \] Lipschitz, bounded polygon/polyhedron
\[ c = c(x) \] piecewise constant on a fixed, finite polygonal/polyhedral partition \( \{\Omega_i\} \) of \( \Omega \)
\[ f \in L^2(Q), \quad v_0 \in L^2(\Omega), \quad \sigma_0 \in L^2(\Omega)^d \]

Find \((v, \sigma)\) such that
\[
\begin{align*}
\nabla v + \frac{\partial \sigma}{\partial t} &= 0, \\
\nabla \cdot \sigma + c^{-2} \frac{\partial v}{\partial t} &= f
\end{align*}
\]
in \( Q \)
\[ v(\cdot, 0) = v_0, \quad \sigma(\cdot, 0) = \sigma_0 \]
on \( \Omega \)
\[ v = 0 \]
on \( \partial \Omega \times [0, T] \)

2nd order wave equation (provided that \( \sigma_0 \) is a gradient)

\[
\begin{align*}
v &= \partial_t U \\
\sigma &= -\nabla U \\
\end{align*}
\]
\[
\begin{align*}
-\Delta U + c^{-2} \partial_{tt} U &= f \quad \text{in } \Omega \\
+ \text{ initial/boundary conditions}
\end{align*}
\]

\[ U \in C^0([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap H^2(0, T; H^{-1}(\Omega)) \]

[Dautray, Lions, 1992]
\[ \nabla v + \frac{\partial \sigma}{\partial t} = 0, \quad \nabla \cdot \sigma + c^{-2} \frac{\partial v}{\partial t} = f \quad \text{in } Q \]

\[ \mathcal{L}_{\text{wave}}(v, \sigma) = (f, 0) \]

multiply by test functions \( \tau \) and \( w \), respectively, and integrate by parts in \( Q = \Omega \times (0, T) \):

\[ \int_Q \left( \nabla v + \frac{\partial \sigma}{\partial t} \right) \cdot \tau \, dV + \int_Q \left( \nabla \cdot \sigma + c^{-2} \frac{\partial v}{\partial t} \right) w \, dV = \int_Q f \, w \, dV \]

\[ \rightarrow \]

**space-time variational formulation**

\[ - \int_Q \left[ v \left( \nabla \cdot \tau + c^{-2} \frac{\partial w}{\partial t} \right) + \sigma \cdot \left( \nabla w + \frac{\partial \tau}{\partial t} \right) \right] \, dV + \int_{\Omega \times \{T\}} (\sigma \cdot \tau + c^{-2} v w) \, dx \]

\[ = \int_Q f \, w \, dV + \int_{\Omega \times \{0\}} (\sigma_0 \cdot \tau + c^{-2} v_0 w) \, dx \]

- \( \nabla v + \frac{\partial \sigma}{\partial t} = 0 \) holds in \( C^0([0, T]; H_0(\text{div}; \Omega)^*) \)

- \( \nabla \cdot \sigma + c^{-2} \frac{\partial v}{\partial t} = f \) holds in \( L^2(0, T; H^{-1}(\Omega)) \)

- \( v = 0 \) on \( \partial \Omega \times [0, T] \) is imposed weakly (details in [3])
Space-time discontinuous Galerkin discretization

\[ \nabla v + \frac{\partial \sigma}{\partial t} = 0, \quad \nabla \cdot \sigma + c^{-2} \frac{\partial v}{\partial t} = f \quad \text{in } Q = \Omega \times (0, T) \]

- introduce a polytopic \textit{space-time mesh} \( T_h = \{ K \} \) of \( Q \), with \( c \) constant in each element
- multiply by test functions and integrate by parts \textit{element by element}
- discretize \((v, \sigma)\) and \((w, \tau)\) in \textit{discontinuous, piecewise polynomial spaces} \( V_p(T_h) \)
- replace interelement traces by \textit{numerical fluxes}

\begin{align*}
\text{elemental DG formulation} \\
- \int_K \left[ v_h \left( \nabla \cdot \tau_h + c^{-2} \frac{\partial w_h}{\partial t} \right) + \sigma_h \cdot \left( \nabla w_h + \frac{\partial \tau_h}{\partial t} \right) \right] \, dV \\
+ \int_{\partial K} \left[ (\hat{v}_h \tau_h + \hat{\sigma}_h w_h) \cdot n^x_K + \left( \hat{\sigma}_h \cdot \tau_h + c^{-2} \hat{v}_h w_h \right) n^t_K \right] \, dS = \int_K f \, w_h \, dV
\end{align*}

where \((n^x_K, n^t_K) \in \mathbb{R}^{d+1}\) denotes the unit normal vector to \( \partial K \) pointing outside \( K \)

\begin{align*}
\text{global DG formulation} \\
\text{add over all } K \in T_h \rightarrow \quad A_{DG}(v_h; \sigma_h, w_h; \tau_h) = \ell_{DG}(w_h, \tau_h)
\end{align*}
Assumption on the meshes

assumption on $\mathcal{T}_h$

each internal face $F$ is either

- **space-like:** $c|\mathbf{n}_F^x| < n_F^t$ ($F \subset \mathcal{F}_h^{\text{space}}$), or
- **time-like:** $n_F^t = 0$ ($F \subset \mathcal{F}_h^{\text{time}}$)

$$\mathcal{F}_h^0 := \Omega \times \{0\}, \quad \mathcal{F}_h^T := \Omega \times \{T\}$$
$$\mathcal{F}_h^\partial := \partial \Omega \times (0, T)$$
Assumption on the numerical fluxes

- across **space-like** faces, the information is transported from $t^-$ to $t^+$
- across **time-like** faces, there is no directionality

\[ \hat{v}_h := \begin{cases} v_h^- & \text{on } \mathcal{F}^0_h \\ \{v_h\} + \beta [\sigma_h]_N & \text{on } \mathcal{F}^{\text{space}}_h \cup \mathcal{F}^T_h \end{cases} \]

\[ \hat{\sigma}_h := \begin{cases} \sigma_h^- & \text{on } \mathcal{F}^{\text{space}}_h \cup \mathcal{F}^T_h \\ \{\sigma_h\} + \alpha [v_h]_N & \text{on } \mathcal{F}^{\text{time}}_h \end{cases} \]

\[ \sigma_0 \quad \text{on } \mathcal{F}^0_h \]

\[ \sigma_h - \alpha v n^x_\Omega \quad \text{on } \mathcal{F}^{\partial}_h \]

\[ \alpha, \beta \in L^\infty(\mathcal{F}^{\text{time}}_h \cup \mathcal{F}^{\partial}_h); \quad \alpha = \beta = 0 \quad [\text{Egger \& al., 2014}] \quad \alpha \beta \geq \frac{1}{4} \quad [\text{Monk, Richter, 2005}] \]
Assumption on the approximation spaces

recall the definition of the wave operator $\mathcal{L}_{\text{wave}}(w, \tau) := \left( \nabla \cdot \tau + c^{-2} \frac{\partial w}{\partial t}, \nabla w + \frac{\partial \tau}{\partial t} \right)$

**assumption on $V_p(\mathcal{T}_h)$**

for all $(w_h, \tau_h) \in V_p(\mathcal{T}_h)$, $\mathcal{L}_{\text{wave}}(w_h, \tau_h) \in V_p(\mathcal{T}_h)$

this is satisfied, e.g., if the restriction of $V_p(\mathcal{T}_h)$ to each mesh element is made of

- total degree space-time polynomials $\mathbb{P}_x^p, \mathbb{P}_t^p$,
- tensor product (in time) polynomials $\mathbb{P}_x^p \times \mathbb{P}_t^p$,
- Trefftz polynomials $\mathcal{L}_{\text{wave}}(w_h, \tau_h) = (0, 0)$
Well-posedness

- case of tensor product (in time) meshes

**key property: coercivity in seminorm**

\[ \mathcal{A}_{DG}(v_h, \sigma_h; v_h, \sigma_h) = |(v_h, \sigma_h)|_{DG}^2 \]

**DG seminorm**

\[
|(w, \tau)|_{DG}^2 = \frac{1}{2} \left| c^{-1} [w]_t \right|_{L^2(F_{h}^{\text{space}})}^2 + \frac{1}{2} \left| [\tau]_t \right|_{L^2(F_{h}^{\text{space}})}^2 + \left| \frac{1}{2} [w]_N \right|_{L^2(F_{h}^{\text{time}})}^2 + \left| \frac{1}{2} \tau \right|_{L^2(F_{h}^{\text{time}})}^2 + \left| \frac{1}{2} w \right|_{L^2(F_{h}^{\partial})}^2
\]

by adapting [Monk, Richter, 2005], one deduces well-posedness, with **no condition on** \( h_t \)

*The case of general, admissible meshes requires minor, technical changes.*
Proof of well-posedness

\((\ast)\) \(A_{DG}(v_h, \sigma_h; w_h, \tau_h) = 0 \quad \forall (w_h, \tau_h) \in V_p(\mathcal{T}_h) \implies (v_h, \sigma_h) = (0, 0)\)

i) take \((w_h, \tau_h) = (v_h, \sigma_h) \rightarrow 0 = A_{DG}(v_h, \sigma_h; v_h, \sigma_h) = |(v_h, \sigma_h)|^2_{DG} \implies \text{all jumps and boundary traces of } v_h \text{ and } \sigma_h \text{ are zero}

ii) using i) in \((\ast)\) and integrating by parts give

\[
\int_K \left[ \left( \nabla \cdot \sigma_h + c^{-2} \frac{\partial v_h}{\partial t} \right) w_h + \left( \nabla v_h + \frac{\partial \sigma_h}{\partial t} \right) \cdot \tau_h \right] \, dV = 0 \quad \forall (w_h, \tau_h) \in V_p(\mathcal{T}_h)
\]

iii) take \(w_h = \uparrow\) and \(\tau_h = \uparrow\) \implies \((v_h, \sigma_h)\) solves the homogeneous wave problem in each \(K\)

iv) steps i) and iii) imply that \((v_h, \sigma_h) = (0, 0)\)
case of tensor product (in time) meshes

Assume that all the traces of the analytical solution on mesh faces are in $L^2$ → error bound in the $L^2$ norm in space at every discrete time $t_n$:

$$
\| c^{-1}(v - v_h) \|_{L^2(\Omega \times \{t_n\})} + \| \sigma - \sigma_h \|_{L^2(\Omega \times \{t_n\})} \leq |(v, \sigma) - (v_h, \sigma_h)|_{\text{DG}(Q_n)} \\
\lesssim |(v, \sigma) - \Pi(v, \sigma)|_{\text{DG}+} \\
\in \mathbf{V}_p(T_h)
$$

(proven by restricting to partial space-time cylinders $Q_n = \Omega \times (0, t_n)$)

Projector $\Pi$

- total degree space-time polynomials $\mathbb{P}^p_{x,t}$: construction in [Monk, Richter, 2005]
- tensor product (in time) polynomials $\mathbb{P}^p_x \times \mathbb{P}^p_t$: $L^2$ projection [3]
- Trefftz polynomials: best approximation [1]

Error estimates (with no condition on $h_t$)
Trefftz finite element spaces (case \( f = 0 \))

recall: \( \mathcal{L}_{\text{wave}}(w, \tau) = \left( \nabla \cdot \tau + c^{-2} \frac{\partial w}{\partial t}, \nabla w + \frac{\partial \tau}{\partial t} \right) \)

Trefftz spaces

continuous spaces

\[
T(K) := \{ (w, \tau) \in H^1(K) : \mathcal{L}_{\text{wave}}(w, \tau) = (0, 0) \}
\]

\[
T(T_h) := \left\{ (w, \tau) \in H^1(T_h)^{1+d} : (w, \tau)|_K \in T(K) \quad \forall K \in T_h \right\}
\]

discrete spaces

\( V_p(K) \subset T(K), \quad V_p(T_h) \subset T(T_h) \)

in each element \( K \), the linear operator \( \mathcal{L}_{\text{wave}} \) is

- homogeneous (= all terms are derivatives of the same order)
- with constant coefficients

\[ \Rightarrow \text{Taylor polynomials of (smooth) functions in ker}(\mathcal{L}_{\text{wave}}) \text{ are in ker}(\mathcal{L}_{\text{wave}}) \]

therefore, we can choose \( V_p(K) \subset T(K) \)

- as a subspace of the polynomial space \( \mathbb{P}^p(K)^{1+d} \)
- with the same order of approximation in \( h \) as \( \mathbb{P}^p(K)^{1+d} \) for functions in \( \text{ker}(\mathcal{L}_{\text{wave}}) \)
Trefftz approximation spaces \((f = 0)\)

Example:

\[
\mathbb{T}^p(K) := \left\{ u \in \mathbb{P}^p(K) : -\Delta u + c^{-2} \frac{\partial^2 u}{\partial t^2} = 0 \right\} \quad \mathbb{V}_p(K) := \left( \frac{\partial \mathbb{T}^{p+1}(K)}{\partial t}, -\nabla(\mathbb{T}^{p+1}(K)) \right)
\]

- reduction of number of degrees of freedom to that of a \(d\)-dimensional problem

<table>
<thead>
<tr>
<th>(d + 1)</th>
<th>Trefftz polyn. (\mathbb{T}^p(K))</th>
<th>full polyn. (\mathbb{P}^p(K))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 1</td>
<td>2(p + 1)</td>
<td>(\frac{1}{2}(p + 1)(p + 2))</td>
</tr>
<tr>
<td>2 + 1</td>
<td>((p + 1)^2)</td>
<td>(\frac{1}{6}(p + 1)(p + 2)(p + 3))</td>
</tr>
<tr>
<td>3 + 1</td>
<td>(\frac{1}{6}(p + 1)(p + 2)(2p + 3))</td>
<td>(\frac{1}{24}(p + 1)(p + 2)(p + 3)(p + 4))</td>
</tr>
<tr>
<td></td>
<td>(\mathcal{O}(p^d))</td>
<td>(\mathcal{O}(p^{d+1}))</td>
</tr>
</tbody>
</table>

\[
\dim(\mathbb{T}^p(K)) = \mathcal{O}(p^d) \ll \dim(\mathbb{P}^p(K)) = \mathcal{O}(p^{d+1})
\]

- same orders of approximation in \(h\) as with the full polynomial spaces
Trefftz approximation spaces \((f = 0)\)

\[ d = 1, \text{ smooth solution, Cartesian mesh; Trefftz (blue) and } \mathbb{Q}_p \text{ (orange) polynomials} \]

\[ p \text{-version: error (in } L^2(\Omega \times \{T\})\text{)) vs. polynomial degree (left) and number of dofs (right)} \]
in $\mathbf{T}(\mathcal{T}_h)$, the DG seminorm is actually a norm

existence and uniqueness of solutions follow from $A_{DG}(v_h, \sigma_h; v_h, \sigma_h) = |(v_h, \sigma_h)|_{DG}^2$

error bounds in the (spatial) $L^2$ norm on space-like interfaces (e.g. on $\Omega \times \{t_n\}$) and in DG norm also follow

error bounds in a global, mesh-independent norm ($L^2(Q)$, in the best case scenario*) have also been proven in [1] by a modified duality argument from [Monk, Wang, 1999]

piecewise smooth coefficients: space-time quasi-Trefftz DG method

[Imbert-Gérard, Moiola, Stocker, 2020]

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* i.e. for $d = 1$ or $d > 1$ and no time-like faces (for impedance b.c.) ; in $H^{-1}(0, T; L^2(\Omega)) \times L^2(0, T; H^{-1}(\Omega)^d)$ for tensor product elements (with Dirichlet b.c.)
PDE-driven, front-advancing mesh construction technique

- progressively advancing in time and stacking tent-pitched objects on top of each other
- each tent is union of \((d + 1)\)-dimensional simplices
- the high of each tent (local advancement in time) is chosen so that the casuality constraint of the PDE is respected (local CFL condition)

→ the PDE is explicitly solvable within each tent

Tent pitching & Trefftz

Ilaria Perugia

Nonstandard finite elements for wave problems
Tent pitching & Trefftz

nonstandard finite elements for wave problems

characteristic speed $\frac{1}{c}$

advancing front
Trefftz: solution of local problems [1], [2]; for an interior tent:

\[
\int_{\partial K^{\text{top}}} \left( (v_h \tau_h + \sigma_h w_h) \cdot n^x_K + \left( \sigma_h \cdot \tau_h + c^{-2} v_h w_h \right) n^t_K \right) \, dS = -\int_{\partial K^{\text{bot}}} \left( (v_{h}^{\text{bot}} \tau_h + \sigma_h^{\text{bot}} w_h) \cdot n^x_K + \left( \sigma_h^{\text{bot}} \cdot \tau_h + c^{-2} v_{h}^{\text{bot}} w_h \right) n^t_K \right) \, dS
\]

- mapping + RK or Taylor

[Ilaria Perugia, Nonstandard finite elements for wave problems]
the solution within these two tents can be evolved in parallel
Tent pitching & Trefftz

\[ T \]

\[ t \]

\[ x \]
Tent pitching & Trefftz
\[ d = 2 \text{ and refined mesh towards a corner} \]
Numerical results: Trefftz + tent pitching

\[ d = 3, \text{ smooth solution, Trefftz on tent-pitched meshes; } h\text{- and } p\text{-version} \]

<table>
<thead>
<tr>
<th>Error</th>
<th>( h )</th>
<th>( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{-6} )</td>
<td>0.95</td>
<td>1</td>
</tr>
<tr>
<td>( 10^{-5} )</td>
<td>0.63</td>
<td>2</td>
</tr>
<tr>
<td>( 10^{-4} )</td>
<td>0.33</td>
<td>3</td>
</tr>
<tr>
<td>( 10^{-3} )</td>
<td>0.25</td>
<td>4</td>
</tr>
</tbody>
</table>

convergence of order \( p + 1 \) in \( h \) (left) and exponential convergence in \( p \) (right)
$d = 2$, smooth solution, Trefftz; tensor product (in time) meshes and tent-pitching
Numerical results: Trefftz + tent pitching

\( d = 2, \) singular solution, Trefftz on tent-pitched meshes

\[
U(r, \varphi, t) = \cos(10t) \sin(\nu \varphi) J_\nu(10r)
\]

\( \nu = \frac{2}{3} \quad \rightarrow \quad U \in H^{\frac{5}{3} - \epsilon}(Q) \)

\[
p = 3; \text{ spatial mesh at } t = 0: \text{ uniform (blue) or with corner refinement (orange)}
\]
Regularity theory in 2D

[S] := \{c_j, j = 1, \ldots, M\} set of all vertices of \{\Omega_i\}, in which \(c\) is piecewise constant

acoustic waves exhibit conical singularities at \(S\): regularity results are given in weighted Sobolev spaces in \(\Omega\) with weight function

\[
\Phi_\delta(x) = \prod_{j=1}^{M} |x - c_j|^\delta_j, \quad \delta_j \in [0, 1)
\]

e.g. \(|u|_{H^{1,1}_\delta(\Omega)} := \|\Phi_\delta \nabla u\|_{L^2(\Omega)}^2\quad (H^{1,1}_\delta(\Omega) \not\subset H^1(\Omega))\)

(used for the analysis of DG + time-stepping [Müller, Schötzau, Schwab, 2018]).

Example: if \(v_0, u_0 \in C_0^\infty(\Omega), \sigma_0 = -\nabla u_0, f \in C_0^\infty(Q), \exists \delta \in [0, 1)^M\) such that \(\forall k_t, k_x \in \mathbb{N},\)

\[
v \in C^{k_t-1}([0, T]; H^{k_x+1,2}_\delta(\Omega)) \quad \sigma \in C^{k_t}([0, T]; H^{k_x,1}_\delta(\Omega))^2
\]

[ Müller, 2017]
Tensor product meshes

- time mesh: \( \mathcal{T}^t_{h_t} \) partition of \((0, T)\) into \(N\) intervals \(I_n\)
- spatial meshes: for each \(1 \leq n \leq N\), \( \mathcal{T}^x_{h_{x,n}} \) shape-regular mesh of \(\Omega\)
  - with non-degenerating faces
  - aligned with \(\{\Omega_i\}\)
  - each mesh element touches at most one element of \(S\)
- space-time mesh: \( \mathcal{T}_h := \mathcal{T}_h(Q) := \{K = K_x \times I_n : K_x \in \mathcal{T}^x_{h_{x,n}}, 1 \leq n \leq N\} \)

Abstract error analysis (Galerkin error \(\lesssim L^2\) projection error)

The critical solution regularity is the regularity in space of \(\sigma\) close to any \(c \in S\):

- if \(F\) is a time-like face of an element \(K\) adjacent to a corner \(c\),
  - then \(\sigma|_F \in L^1(F)^2\), not necessarily \(L^2(F)^2\)

\[ \rightarrow \] modify the DG seminorm and apply Hölder in \(L^1-L^\infty\) (instead of Cauchy-Schwarz)

[@Wihler, 2002]
Tensor product meshes

**mesh grading in space** (like in the elliptic case)

lack of smoothness $\rightarrow$ loss in the accuracy of the $L^2$ projection of the solution in the elements $K = K_\mathbf{x} \times I_n$ that are close to any $\mathbf{c} \in \mathcal{S}$

a reduction of the size of $K_\mathbf{x}$ depending on

- the corner weight $\delta_{\mathbf{c}}$
- the polynomial approximation degree $p_K$

can restore the largest possible convergence rates

suitable graded spatial meshes $\mathcal{T}_{h_\mathbf{x}}$ can be constructed from a quasi-uniform initial mesh $\mathcal{T}_{0_\mathbf{x}}$ of $\Omega$ of size $h_\mathbf{x}$ by $J$ levels of local bisection refinement ($J = J(h_\mathbf{x}, \delta_{\mathbf{c}}, p_K)$)

[Gaspoz, Morin, 2009]

- uniform, $h_\mathbf{x} = 0.25$
- graded, $p = 1$
- graded, $p = 2$
Error bounds on locally refined meshes

assume, for simplicity, constant $c$, uniform $p$

- fix $h_t, h_x > 0$, and construct the uniform mesh $\mathcal{T}_t^{h_t}$ and the locally refined mesh $\mathcal{T}_x^{h_x}$
- on any time-like face $F$, define the numerical flux parameters as $\alpha = \beta^{-1} = \frac{h_x}{c |F_x|}$
- assume that $c h_t \approx h_x$ ($h_x$ is the size of the largest element of $\mathcal{T}_x^{h_x}$)

**error bounds**

for every discrete time $t_n$, we have

$$
\| c^{-1}(v - v_h) \|_{L^2(\Omega \times \{t_n\})} + \| \sigma - \sigma_h \|_{L^2(\Omega \times \{t_n\})^2} \leq |(v, \sigma) - (v_h, \sigma_h)|_{DG(\mathcal{Q}_n)} \lesssim h^{p + \frac{1}{2}}
$$

(same convergence rates as for smooth solutions)

**Remark:** $\dim(\mathbf{V}(\mathcal{T}_h)) = \mathcal{O}(h^{-3})$ (like for a (2 + 1)-dimensional elliptic problem)

**Q:** Can we obtain the same convergence rates with $\mathcal{O}(h^{-2})$ degrees of freedom? (like for a 2-dimensional elliptic problem)
the assumption $c h_t \simeq h_x$ is necessary to obtain the highest convergence rates

stability of the DG formulation and best approximation-type estimates are valid with no condition on $h_t$

the solutions obtained with anisotropic (in time) space-time meshes are not accurate, still they contain meaningful information

$\mathcal{T}_{(0,0)}$ coarsest space-time mesh
$\mathcal{T}_{(L,L)}$ finest space-time mesh (red)
$\mathcal{T}_{(l_x,l_t)}$ intermediate meshes

$w(l_x,l_t) := (v(l_x,l_t), \sigma(l_x,l_t))$ solution on $\mathcal{T}_{(l_x,l_t)}$

$w_F := w_{(L,L)}$ full space-time solution

$w_S := \sum_{l=0}^{L} w(l,L-l) - \sum_{l=1}^{L} w(l-1,L-l)$ “sparse” solution

Count of degrees of freedom ($h$-version): 

# d.o.f.s for $w_F \lesssim 2^{3L} = \mathcal{O}(h_L^{-3})$

# d.o.f.s for $w_S \lesssim 2^{2L} = \mathcal{O}(h_L^{-2})$

($\simeq$ one time-step on the finest spatial mesh)
Numerical results: combination formula

Expected convergence rates: full $\mathcal{O}(\text{N dofs})^{-\frac{p+1/2}{3}}$, sparse $\mathcal{O}(\text{N dofs})^{-\frac{p+1/2}{2}}$

$p = 1$, full (blue), sparse (red)
smooth solution, uniform meshes (left), conical singularity, spatially graded meshes (right)

Obtained convergence rates: full $\mathcal{O}(\text{N dofs})^{-\frac{p+1}{3}}$, sparse $\mathcal{O}(\text{N dofs})^{-\frac{p+1}{2}}$
Numerical results: combination formula

\[ O(N_{dofs})^{\frac{p+1/2}{3}}, \text{ sparse } O(N_{dofs})^{\frac{p+1/2}{2}} \]

\( p = 2 \), full (blue), sparse (red)
smooth solution, uniform meshes (left), conical singularity, spatially graded meshes (right)

Obtained convergence rates: full \( O(N_{dofs})^{\frac{p+1}{3}} \), sparse \( O(N_{dofs})^{\frac{p+1}{2}} \)
Thank you for your attention!