Positive harmonic functions
on the Heisenberg group

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MAIN QUESTION
Classify the positive harmonic functions on nilpotent groups.

Organization

1. Harmonic functions.
2. The Heisenberg group.
3. The partition function.
4. Abelian groups.
5. Nilpotent groups.
6. Inducing harmonic characters.
7. Classification on the Heisenberg group

1. Harmonic functions

Let G be a group generated by a finite set S and $\mu = \sum_{s\in S} \mu_s \delta_s$ be a positive measure with support S.

Definition $\mathcal{H}^+_\mu = \{h : G \to [0,\infty[\,,\ P_\mu h = h\}$ = convex cone of positive μ-harmonic functions.
Here $P_\mu h(g) = \sum_{s\in S} \mu_s\, h(sg)$.
We set $\mathcal{E}_\mu$ for the set of extremal h in $\mathcal{H}^+_\mu$.

Choquet, 1950 For all f in $\mathcal{H}^+_\mu$, there exists a positive measure ν on $\mathcal{E}_\mu$ such that $f = \int_{\mathcal{E}_\mu} h\, d\nu(h)$.

History
The positive μ-harmonic functions were described
⋆ on abelian groups by Choquet and Deny in the 1950s.
And, if the semigroup G^+_μ spanned by S is equal to G,
⋆ on nilpotent groups by Margulis in the 1960s,
⋆ on hyperbolic groups by Ancona in the 1980s.

2. The Heisenberg group

The aim of the talk is to describe $\mathcal{E}_\mu$ for the group

$$G = H_3(\mathbb{Z}) = \{g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x,y,z \in \mathbb{Z}\}.$$

Writing $g = (x,y,z)$, the product is
$(x,y,z)\,(x',y',z') = (x{+}x', y{+}y', z{+}z'{+}xy')$.

Today's Main Theorem When $G = H_3(\mathbb{Z})$, every h in $\mathcal{E}_\mu$ is either a character or induced of a character.

To be concrete, we choose the south-west measure $\mu_{SW} = \delta_{a^{-1}} + \delta_{b^{-1}}$ with $a = (1,0,0)$, $b = (0,1,0)$.
h harmonic means $h(g) = h(a^{-1}g) + h(b^{-1}g)$ i.e.

$$h(x,y,z) = h(x-1, y, z-y) + h(x, y-1, z).$$

Note that the functions $\widetilde{h} = 2^{-x-y} h$ are the $\widetilde{\mu}$-harmonic functions for the probability measure $\widetilde{\mu} = \frac{1}{2}\mu_{SW}$.

3. The partition function

First look for functions h that do not depend on x:

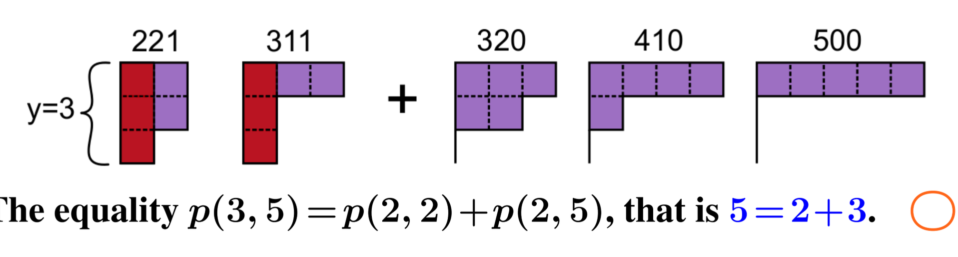
$$h(y,z) = h(y, z-y) + h(y-1, z).$$

A solution is the partition function introduced by Cayley and Sylvester in 1850:
$p(y,z)$ is the number of partitions of z in y integers:

$|\{n_1 \geq \cdots \geq n_y \geq 0 \text{ with } n_1 + \cdots + n_y = z\}|$.

One has $p(y,z) = p(y, z-y) + p(y-1, z)$.

Proof for $p(3,5)$ Partitions of height 3 and area 5.



The equality $p(3,5) = p(2,2) + p(2,5)$, that is $5 = 2+3$. ○

↑ z								
0	0	1	4	7	9	10	11	
0	0	1	3	5	6	7	7	
0	0	1	3	4	5	5	5	
0	0	1	2	3	3	3	3	
0	0	1	2	2	2	2	2	
0	0	1	1	1	1	1	1	
0	(1)	1	1	1	1	1	1	→ y
0	0	0	0	0	0	0	0	

the diagonal p(z,z) is the partition function of Hardy-Ramanujan.

Remark: since $(x,y,z) \mapsto (y, x, xy-z)$ is a group morphism, the function $p(x, xy-z)$ is also harmonic.

4. Abelian groups

Now look for functions that do not depend on z.
The function h is then harmonic on $\mathbb{Z}^2$:

$$h(x,y) = h(x-1, y) + h(x, y-1).$$

Choquet-Deny, 1950 For G abelian, every h in $\mathcal{E}_\mu$ is proportional to a character χ.

i.e. $\chi(gg') = \chi(g)\chi(g')$ and $\sum_{s\in S} \mu_s \chi(s) = 1$.

Short proof The equality $h(g) = \sum_s \mu_s h(sg)$ decomposes h as a sum of harmonic functions. Since h is extremal, one must have $h(sg) = \chi(s)h(g)$. ○

5. Nilpotent groups

Margulis, 1960 For G nilpotent and $G^+_\mu = G$, every h in $\mathcal{E}_\mu$ is proportional to a character χ.

Proof Let $c = aba^{-1}b^{-1}$ be in the center of G. One can assume that a, b, a^{-1}, b^{-1}, c belong to S.
As above one has $h(cg) = q\, h(g)$ with $q > 0$.
We want to prove that $q = 1$. We write

$$\begin{aligned} h(g) &= P^{4n}_\mu h(g) \geq \alpha^{4n} h(a^n b^n a^{-n} b^{-n} g) \\ &= \alpha^{4n} h(c^{n^2} g) = \alpha^{4n} q^{n^2} h(g). \end{aligned}$$

Hence $q \leq 1$. Similarly $q^{-1} \leq 1$.
By induction on the rank of G, h is a character. ○

The harmonic function $p(y,z)$ is not a character!

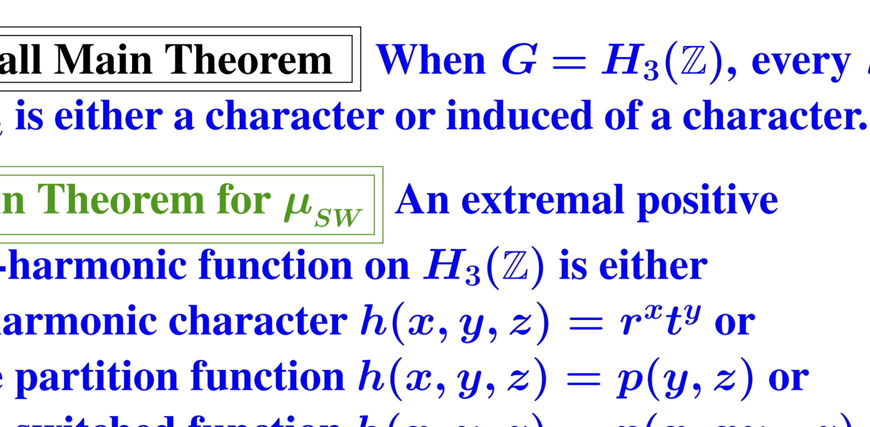
6. Inducing harmonic characters

Let $S_0 \subset S$ be a maximal abelian subset, $\mu_0 = \mu|_{S_0}$ and $G_0 = \langle S_0 \rangle$.
Let χ_0 be a μ_0-harmonic character on a left G_0-orbit in G, extended by 0 to G.

Lemma The sequence $P^n_\mu \chi_0$ is non-decreasing. If its limit h_{G_0,χ_0} is finite, it is μ-harmonic and extremal.

Example For $\mu = \mu_{SW}$, $G_0 = a^{\mathbb{Z}}$ and $\chi_0 = 1_{G_0}$, one has $h_{G_0,\chi_0}(x,y,z) = p(y,z)$.

Proof of Example $P^n_\mu \chi_0(g) = \sum_{w\in\{a,b\}^n} \chi_0(w^{-1}g)$ is the number of words $w \in \{a,b\}^n$ with $g \in wG_0$, i.e. the number of partitions of height y and area z.



Example: $w = aababa$ reaches $g = (x,2,5)$. ○

Recall Main Theorem When $G = H_3(\mathbb{Z})$, every h in $\mathcal{E}_\mu$ is either a character or induced of a character.

Main Theorem for μ_{SW} An extremal positive μ_{SW}-harmonic function on $H_3(\mathbb{Z})$ is either
⋆ a harmonic character $h(x,y,z) = r^x t^y$ or
⋆ the partition function $h(x,y,z) = p(y,z)$ or
⋆ the switched function $h(x,y,z) = p(x, xy-z)$
(or a right translate of a multiple of one of these).

7. Harmonic functions on the Heisenberg group

Proof of Main Theorem for μ_{SW}

An extremal harmonic function h either is a character or is induced of a character.

We know that $h(g) = \sum_{w\in A^n} h(w^{-1}g)$ with $A = \{a,b\}$.

Case 1 If $\lim_{n\to\infty} h(a^{-n}g) \not\equiv 0$, h is induced from $a^{\mathbb{Z}}$.

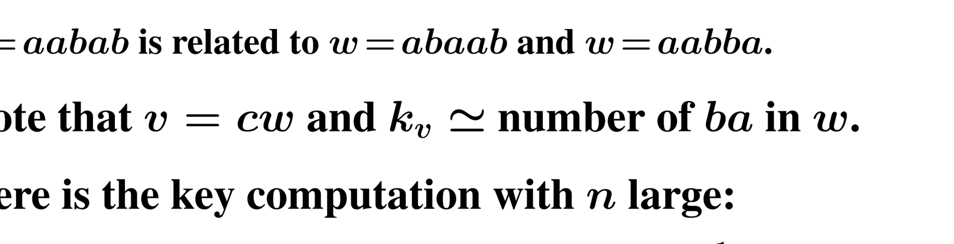
This sequence is decreasing, its limit χ_0 is a-invariant, and the induced of χ_0 is bounded by h. ○

Case 1' If $\lim_{n\to\infty} h(b^{-n}g) \not\equiv 0$, h is induced from $b^{\mathbb{Z}}$.

Same proof. ○

Case 2 If these two limits are 0, h is a character.

We check that h is c-invariant with $c = (0,0,1)$. Set
$\mathcal{R}_n := \{(v,w) = (u_0abu_1, u_0bau_1) \in A^n \times A^n\}$.
k_v = number of occurences of the subword ab in v.



$v = aabab$ is related to $w = abaab$ and $w = aabba$.

Note that $v = cw$ and $k_v \simeq$ number of ba in w.

Here is the key computation with n large:

$$\begin{aligned} h(cg) &= \sum_{v\in A^n} h(v^{-1}cg) \simeq \sum_{(v,w)\in\mathcal{R}_n} \frac{h(w^{-1}g)}{k_v} \\ &\simeq \sum_{w\in A^n} h(w^{-1}g) = h(g). \end{aligned}$$

Letting n go to ∞, this proves $h(cg) = h(g)$.
The function h is c-invariant, hence a character. ○

To justify this computation, we need

Lemma In Case 2, $\lim_{n\to\infty} \sum_{\substack{v\in A^n \\ k_v \leq k_0}} h(v^{-1}g) \equiv 0$, for all $k_0 \geq 1$.

The proof of this lemma is by induction on k_0. ○

For more, see: Journal de l'École Polytechnique 8 (2021) p.973-1003.