Finite groups of birational transformations

Yuri Prokhorov

Steklov Mathematical Institute

8th European Congress of Mathematics
Portorož, Slovenia

$\mathbb{k}$ is a field of characteristic 0; typically $\mathbb{k} = \mathbb{C}$

Cremona group

Two equivalent definitions:

- $\mathrm{Cr}_n(\mathbb{k}) = \mathrm{Aut}_{\mathbb{k}}\, \mathbb{k}(x_1, \ldots, x_n) = \mathrm{Aut}_{\mathbb{k}}\{\text{purely transcendental extension}\}$
- $\mathrm{Cr}_n(\mathbb{k}) = \mathrm{Bir}(\mathbb{P}^n) = \{\text{ birational transformations of } \mathbb{P}^n\} = \mathrm{Bir}(\mathbb{A}^n)$

Easy facts:

- $\mathrm{Cr}_1(\mathbb{k}) = \mathrm{PGL}_2(\mathbb{k})$,
- $\mathrm{Cr}_n(\mathbb{k})$ is infinite dimensional for $n \geq 2$.

Example

- Quadratic involution $(x, y) \longmapsto (1/x, 1/y)$;
- De Jonquières involution $(x, y) \longmapsto (x, p(x)/y)$,
  $p(x)$ is a polynomial $\deg(f) = 2g + 1$ without multiple factors.

Case $n = 2$. History:

Cremona, Max Noether, S. Kantor, Bertini, Castelnuovo, Enriques, H. Hudson, Iskovskikh, Gizatullin, Cantat, Lamy, Blanc ...

Problem

- *Describe the structure of finite subgroups of* $\mathrm{Cr}_n(\Bbbk)$.
- *Describe the structure of finite subgroups of* $\mathrm{Bir}(X)$, *where* X *is alg. variety.*

G-varieties

Definition (Manin)

Let G be a finite group and let X be an algebraic variety.

- X is a G-variety if it is equipped with a regular faithful action $G \hookrightarrow \mathrm{Aut}(X)$.
- A map $X \dashrightarrow Y$ of G-varieties is a G-map if it commutes with the action.

Notation

$\Bbbk(X)$ is the field of rational functions

Facts

- *For any G-variety X the action $G \hookrightarrow \mathrm{Aut}(X)$ induces $G \hookrightarrow \mathrm{Aut}_{\Bbbk}(\Bbbk(X))$.*
- *Let $\mathbb{K}/\Bbbk$ be finitely generated and let $G \subset \mathrm{Aut}_{\Bbbk}(\mathbb{K})$ be a finite group. Then $\exists$ a G-variety X and an isomorphism $\Bbbk(X) \simeq_{\Bbbk} \mathbb{K}$ inducing $G \subset \mathrm{Aut}_{\Bbbk}(\mathbb{K})$.*

Involutions in $\mathrm{Cr}_2(\Bbbk)$

Theorem (Bertini,. . . , Bayle-Beauville)

Let $\tau \in \mathrm{Cr}_2(\Bbbk)$ be an element of order 2. Then τ is induced by one of the following actions on rational surface X

τ	X *and* τ
linear involution on $\mathbb{P}^2$	$\mathbb{P}^2$
de Jonquières involution	*conic bundle* $(x, y) \to x$ $(x, y) \longmapsto (x, p(x)/y)$
Geiser involution	*del Pezzo surface of degree* 2 $X = \{y^2 = f(x_1, x_2, x_3)\} \subset \mathbb{P}(1, 1, 1, 2)$ *deck involution* $X \to \mathbb{P}^2$
Bertini involution	*del Pezzo surface of degree* 1 $X = \{z^2 = f(x_1, x_2, y)\} \subset \mathbb{P}(1, 1, 2, 3)$ *deck involution* $X \to \mathbb{P}(1, 1, 2)$

G-varieties

Corollary

Let $\mathbb{K}/\Bbbk$ be finitely generated. Then

finite subgroups $G \subset \mathrm{Aut}_{\Bbbk}(\mathbb{K})$ modulo conjugacy	$\stackrel{1:1}{\longleftrightarrow}$	G-varieties X such that $\Bbbk(X) \simeq_{\Bbbk} \mathbb{K}$ modulo G-birational equivalence

Definition

X is rational if $X \underset{\text{bir}}{\sim} \mathbb{P}^n$

Equivalently: $\Bbbk(X)/\Bbbk$ is purely transcendental

Corollary

finite subgroups $G \subset \mathrm{Cr}_n(\Bbbk)$ modulo conjugacy	$\stackrel{1:1}{\longleftrightarrow}$	rational G-varieties modulo G-birational equivalence

G-minimal model program

Theorem (regularization, completion and resolution of singularities)

For any G-variety X there exists a smooth projective G-variety Y that is G-birationally equivalent to X.

Definition

X has $G\mathbb{Q}$-factorial singularities if a multiple of any G-invariant Weil divisor on X is Cartier.

Terminal singularities

- very mild, $\operatorname{codim} \geq 3$,
- the smallest class which is closed under MMP.

Example ($\dim = 3$)

- $\mathbb{A}^3/\boldsymbol{\mu}_r, \quad (x_1, x_2, x_3) \longmapsto (\zeta x_1, \zeta^{-1} x_2, \zeta^a x_3), \quad \gcd(a, r) = 1$
- isolated $\quad f(x_1, x_2, x_3) + x_4 g(x_1, x_2, x_3, x_4) = 0$,
  where f is an equation of a ADE-singularity

G-minimal model program

Definition (for simplicity assume that $\Bbbk$ is uncountable and $\bar{\Bbbk} = \Bbbk$)

X is uniruled for a general point $x \in X$ $\quad\exists\quad$ rational curve $C \subset X$, $x \in C$.
X is rationally connected if general points $x_1, x_2 \in X$ can be connected by a rational curve.

Definition

G-Mori fiber space (G-MFS) is a fibration $f : Y \to Z$ such that

- $0 \le \dim(Z) < \dim(X)$,
- Y has only terminal $G\mathbb{Q}$-factorial singularities,
- $\operatorname{rk}\operatorname{Pic}(Y/Z)^G = 1$,
- $-K_Y$ is f-ample.

Theorem (Birkar, Cascini, Hacon, McKernan 2010)

Let X be an uniruled G-variety. Then $\exists$ birational G-map $X \dashrightarrow Y$ where Y has a structure of G-MFS $\quad f : Y \to Z$.

Cremona group of rank 2

MMP in dimension 2 (Manin, Iskovskikh)

- terminal = smooth, $X \overset{G\text{-MMP}}{\dashrightarrow} Y$ is a composition of blowups of G-orbits
- Mori fiber space $f : Y \to Z$
  - Z is a point $\Longrightarrow$ Y is a del Pezzo surface with $\operatorname{rk} \operatorname{Pic}(Y)^G = 1$,
  - Z is a curve $\Longrightarrow$ f is a conic bundle.

Dolgachev & Iskovskikh 2009

- Classification of admissible automorphism groups of del Pezzo surfaces,
- Description of admissible conic bundles.

Theorem (Dolgachev – Iskovskikh)

Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple finite group. Either $G \simeq \mathfrak{A}_5$ or

G	$\lvert G\rvert$	X
$\mathfrak{A}_6$	360	$\mathbb{P}^2$
$\mathrm{PSL}_2(\mathrm{F}_7)$	168	$\mathbb{P}^2$
$\mathrm{PSL}_2(\mathrm{F}_7)$	168	$X = \{y^2 = x_1^3x_2 + x_2^3x_3 + x_3^3x_1\} \subset \mathbb{P}(1,1,1,2)$

Cremona group of rank 3

MMP in dimension 3

- terminal $\implies$ isolated, classified up to analytic equivalence [Reid, Mori],
- $X \xrightarrow{G\text{-MMP}} Y$ is a composition of divisorial contractions and flips
- Mori fiber space $f : Y \to Z$
  - Z is a point $\implies$ Y is a (singular) Fano 3-fold with $\operatorname{rk}\operatorname{Pic}(Y)^G = 1$,
  - Z is a curve $\implies$ f is a del Pezzo fibration,
  - Z is a surface $\implies$ f is a conic bundle.

Main difficulty

Classification of Fano 3-folds with terminal singularities is not known.

Theorem (Prokhorov 2012)

Let $G \subset \mathrm{Cr}_3(\mathbb{C})$ be a simple finite group. Then G is isomorphic to one of the following:

$$\mathfrak{A}_5, \quad \mathfrak{A}_6, \quad \mathfrak{A}_7, \quad \mathrm{PSL}_2(\mathrm{F}_7), \quad \mathrm{SL}_2(\mathrm{F}_8), \quad \mathrm{PSp}_4(\mathrm{F}_3).$$

All the possibilities occur.

Theorem (Prokhorov 2012)

Let X be a rationally connected 3-fold and let $G \subset \mathrm{Bir}(X)$ be a simple finite group. If $G \not\subset \mathrm{Cr}_2(\mathbb{C})$, then

G	X	*rational ?*
$\mathfrak{A}_7$	$X_6' = \{\sigma_1^7 = \sigma_2^7 = \sigma_3^7 = 0\} \subset \mathbb{P}^5 \subset \mathbb{P}^6$	*NO*
	$\sigma_d^k(x_1, \ldots, x_k)$ *are elementary symmetric polynomials*	
$\mathfrak{A}_7$	$\mathbb{P}^3$	*YES*
$\mathrm{PSp}_4(\mathrm{F}_3)$	$\mathbb{P}^3$	*YES*
$\mathrm{PSp}_4(\mathrm{F}_3)$	*Burkhardt quartic* $X_4^{\mathrm{b}} = \{\sigma_1^6 = \sigma_4^6 = 0\} \subset \mathbb{P}^4 \subset \mathbb{P}^5$	*YES*
$\mathrm{SL}_2(\mathrm{F}_8)$	*Fano 3-fold* $X_{12}^{\mathrm{m}} \subset \mathbb{P}^8$ *of genus* 7	*YES*
$\mathrm{PSL}_2(\mathrm{F}_{11})$	*Klein cubic* $X_3^{\mathrm{k}} = \{x_1x_2^2 + x_2x_3^2 + \cdots x_5x_1^2 = 0\} \subset \mathbb{P}^4$	*NO*
$\mathrm{PSL}_2(\mathrm{F}_{11})$	*Fano 3-fold* $X_{14}^{\mathrm{a}} \subset \mathbb{P}^9$ *of genus* 8	*NO*

Jordan property

Theorem (C. Jordan)

$\forall n\ \exists\ \mathrm{j}(n)$ *such that for any finite subgroup* $G \subset \mathrm{GL}_n(\mathbb{C})$ $\exists$ *a normal abelian subgroup* $A \subset G$ *of index at most* $\mathrm{j}(n)$.

Theorem (H. Minkowski)

$\forall n\ \exists\ \mathrm{b}(n)$ *such that for every finite subgroup* $G \subset \mathrm{GL}_n(\mathbb{Q})$ *one has* $|G| \le \mathrm{b}(n)$.

Question (J.-P. Serre 2009)

Do these properties hold for Cremona groups?

Definition (V. L. Popov)

- A group Γ is Jordan if $\exists\ \mathrm{j}(\Gamma)$ s. t. $\forall$ finite subgroup $G \subset \Gamma$ has a normal abelian subgroup A of index $[G : A] \le \mathrm{j}(\Gamma)$.
- A group Γ is bounded if $\exists\ \mathrm{b}(\Gamma)$ s. t. $\forall$ finite $G \subset \Gamma$ one has $|G| \le \mathrm{b}(\Gamma)$.

Boundedness property

Theorem (Prokhorov-Shramov 2014 & Birkar's BAB)

Let $\Bbbk$ be a finitely generated field over $\mathbb{Q}$ and let X be a variety over $\Bbbk$. Then $\operatorname{Bir}(X)$ is bounded.

Algebraic form:

Theorem (Positive answer to a question of J.-P. Serre 2010)

Let $\mathbb{K}$ be a finitely generated field over $\mathbb{Q}$. Then the group $\operatorname{Aut}(\mathbb{K})$ is bounded.

Theorem (Birkar 2016)

Fix $d > 0$. The set of all Fano varieties X, $\dim(X) \le d$ with at worst terminal singularities form a bounded family, i.e. they are parameterized by a scheme of finite type.

Jordan property of rationally connected varieties

Theorem (Prokhorov-Shramov 2016 & C. Birkar's BAB)

Let X be rationally connected. Then $\mathrm{Bir}(X)$ *is Jordan.*
Moreover, $\mathrm{Bir}(X)$ *in uniformly Jordan, that is, the constant* $\mathrm{j}(\mathrm{Bir}(X))$ *depends only on* $\dim(X)$.

Weak algebraic form:

Corollary

$\mathrm{Cr}_n(\Bbbk)$ *is Jordan.*

p-groups: Beauville, Prokhorov, Prokhorov-Shramov, O. Haution, Jinsong Xu

Theorem

Let X be rationally connected $\dim(X) = n$.
Let $G \subset \mathrm{Bir}(X)$ be a finite p-subgroup.
If $p > n + 1$, then G is abelian generated by at most n elements.

Jordan property of arbitrary varieties

Example (Yu. Zarhin)

Let C be an elliptic curve and let $X = C \times \mathbb{P}^1$.
Then the group $\operatorname{Bir}(X)$ is not Jordan.

Theorem (V. L. Popov, Yu. Zarhin)

Let X be an algebraic surface.
$\operatorname{Bir}(X)$ *is not Jordan* $\iff X \underset{\text{bir}}{\sim} \mathbb{P}^1 \times C$, *where C is an elliptic curve.*

Theorem (Prokhorov-Shramov)

Let X be an algebraic variety. Then the following assertions hold.

- *X is non-uniruled* $\implies$ $\operatorname{Bir}(X)$ *is Jordan.*
- *X has irregularity $q(X) = 0$* $\implies$ $\operatorname{Bir}(X)$ *is Jordan.*
- *X is non-uniruled and $q(X) = 0$* $\implies$ $\operatorname{Bir}(X)$ *is bounded.*

Jordan property (3-dimensional case)

Definition

The weak Jordan constant $\mathrm{j_w}(\Gamma)$ of Γ is the minimal j such that
$\forall\ G \underset{finite}{\subset} \Gamma$ there exists $A \underset{abelian}{\subset} G$, $[G : A] \le j$ (not necessarily normal).

Fact : $$\mathrm{j_w}(\Gamma) \le \mathrm{j}(\Gamma) \le \mathrm{j_w}(\Gamma)^2$$

Theorem (Prokhorov-Shramov)

X rationally connected 3-fold $\Longrightarrow \mathrm{j_w}(\mathrm{Bir}(X)) \le 10368$
and the bound is sharp.

3-dimensional case

Theorem (Prokhorov-Shramov)

Let $\dim(X) = 3$*. Then* $\mathrm{Bir}(X)$ *is not Jordan* $\iff$ *either*

- $X \underset{\mathrm{bir}}{\sim} C \times \mathbb{P}^2$*, where* C *is an elliptic curve, or*
- $X \underset{\mathrm{bir}}{\sim} S \times \mathbb{P}^1$*, where* S *is one of the following:*
  - *an abelian surface;*
  - *a bielliptic surface;*
  - *a surface with* $\varkappa(X) = 1$ *such that the Jacobian fibration of the pluricanonical fibration* $\phi\colon S \to B$ *is locally trivial.*

Idea of the proof.

X is uniruled, not rationally connected $\xrightarrow{\text{MMP}}$
$\exists\ X \dashrightarrow Z$ with rationally connected fibers, Z is not uniruled, $\dim(Z) = 1$ or 2.

$$(*) \qquad 1 \longrightarrow \operatorname{Bir}(X_\eta) \longrightarrow \operatorname{Bir}(X) \longrightarrow \operatorname{Bir}(Z), \qquad \eta := \operatorname{Spec}(\Bbbk(Z))$$

Case: Z is a curve. $\mathbb{K} = \Bbbk(Z)$, $S := X_\eta$.

Proposition (Prokhorov-Shramov)

Let $\mathbb{K}$ contains all roots of 1, S be a surface over $\mathbb{K}$ s.t. S is not $\mathbb{K}$-rational, S is $\bar{\mathbb{K}}$-rational, and $S(\mathbb{K}) \neq \varnothing$. Then $\operatorname{Bir}(S)$ is bounded.

Case: Z is a surface. Assume: $\operatorname{Bir}(X)$ is not Jordan.

- Bandman–Zarhin: $X \underset{\text{bir}}{\sim} Z \times \mathbb{P}^1$,
- (*) $\Longrightarrow$ $\operatorname{Bir}(Z)$ is not bounded,
- $\varkappa(Z) = 0 \Longrightarrow Z$ is abelian or bielliptic ($\neq$ K3, Enriques),
- $\varkappa(Z) = 1 \Longrightarrow \exists$ elliptic fibration $\phi : Z \longrightarrow C \Longrightarrow$ apply Lang-Néron (=functional Mordell-Weil) theorem to the generic fiber of $\operatorname{Jac}(\phi)$. □

Essential dimension

Definition (Buhler – Reichstein 1997)

Let G be a finite group and let V be its faithful linear representation.
Essential dimension:

$$\mathrm{ed}(G) := \min \left\{ \dim(X) \;\middle|\; \begin{array}{c} \exists \quad V \dashrightarrow X \\ \text{dominant rational} \\ G\text{-map of } G\text{-varieties} \end{array} \right\}$$

Fact

$\mathrm{ed}(G)$ *does not depend on* V

Motivation.

$$\mathrm{ed}(G) \overset{\text{informally}}{=\!=\!=} \min \{\text{parameters needed to describe faithful representation}\}$$

$$\mathrm{ed}(\mathfrak{S}_n) \overset{\text{informally}}{=\!=\!=} \min \left\{ \begin{array}{c} \text{parameters needed to describe general polynomial} \\ \text{of degree } n \text{ modulo Tschirnhaus transformations} \end{array} \right\}$$

Essential dimension

Theorem

- $n - 3 \geq \mathrm{ed}(\mathfrak{S}_n) \geq \mathrm{ed}(\mathfrak{A}_n) \geq 2\lfloor n/4 \rfloor$ *(Buhler – Reichstein)*
- $\mathrm{ed}(\mathfrak{A}_6) = 3$ *(Serre 2008)*
- $\mathrm{ed}(\mathfrak{A}_7) = \mathrm{ed}(\mathfrak{S}_7) = 4$ *(A. Duncan)*

$$\mathrm{ed}(G) \leq \mathrm{rdim}(G) := \min \left\{ \dim(\text{faithful representation}) \right\}$$

Facts

- *If G is a p-group* $\implies$ $\mathrm{ed}(G) = \mathrm{rdim}(G)$ [Karpenko - Merkurjev 2008],
- *In general,* $\mathrm{ed}(G) \neq \mathrm{rdim}(G)$.

Theorem (Reichstein)

$\mathrm{rdim}(G) \leq \mathrm{ed}(G) \cdot \mathrm{j}(\mathrm{ed}(G))$, *where* $\mathrm{j}(n)$ *is the Jordan constant.*

Conjugacy invariants

Question

Let $G,\ G' \subset \mathrm{Bir}(X), \quad G \simeq G'$. *How can one conclude that* G *and* G' *are* not *conjugate?*

$\Updownarrow$

Question

Let X *and* X' *be* G*-varieties. How can one conclude that* X *and* X' *are* not G*-birational?*

Methods.

- Fixed-point locus: non-uniruled component $\{x \mid g(x) = x \quad \forall g \in G\}_{\mathrm{codim}=1}$,
- Maximal singularities method [Iskovskikh, Cheltsov, Shramov],
- Cohomological invariants: $H^1(G,\ Pic(X))$ [Bogomolov-Prokhorov],
- equivariant Burnside group [Hassett, Kresch, Tschinkel]