Boundary theory and amenability: from Furstenberg's Poisson formula to boundaries of Drinfeld doubles of quantum groups

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Harmonic functions

Let G be a locally compact group, μ a probability measure on G. Define

$$P_\mu(f)(g) = \int_G f(gh)d\mu(h),$$

$$H^\infty(G,\mu) = \{f \in L^\infty(G) \mid P_\mu(f) = f\}.$$

The latter is a commutative G-von Neumann algebra with product

$$f_1 \cdot f_2 = \lim_{n\to\infty} P_\mu^n(f_1 f_2) \qquad \text{(pointwise convergence)},$$

its spectrum is the **Poisson boundary** of G (or (G,μ)).

Remark. If G is a real semisimple Lie group, then for a large class of measures the harmonic functions are exactly the solutions of $\Delta f = 0$, where Δ is any left-invariant elliptic second order differential operator such that $\Delta 1 = 0$.

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Poisson integral

Assume G acts on a probability space (X,ν). The measure ν is called μ-stationary if

$$\mu * \nu = \nu.$$

Given such a measure, we have a map

$$\mathscr{P}_\nu \colon L^\infty(X,\nu) \to H^\infty(G,\mu), \quad \mathscr{P}_\nu(f)(g) = \int_X f(gx)\,d\nu(x).$$

Computing the Poisson boundary is equivalent to finding (X,ν) such that $\mathscr{P}_\nu$ is a (complete) order isomorphism.

Furstenberg's Poisson formula

Assume now that G is a connected real semisimple Lie group with finite center, $K \subset G$ a maximal compact subgroup, and μ is a left K-invariant absolutely continuous measure such that $\operatorname{supp} \mu^{*n}$ contains a neighbourhood of the identity for some $n \geq 1$.

Theorem (Furstenberg)

The Poisson boundary of (G, μ) is $(G/H(G), m)$, where $H(G)$ is a unique up to conjugacy maximal cocompact amenable subgroup of G and m is the unique K-invariant probability measure.

Consider the Iwasawa decomposition $G = KAN$.

Moore: we can take

$$H(G) = N_G(AN) = Z_K(A)AN.$$

Examples

1) $G = SL_2(\mathbb{R})$, $K = SO_2(\mathbb{R})$, $AN = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0\right\}$,

$$H(G) = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \neq 0\right\}.$$

Then $G/K \cong \mathbb{H}$, $G/H(G) \cong \mathbb{R} \cup \{\infty\}$ and Furstenberg's theorem gives the usual Poisson formula for $\mathbb{H}$.

2) $G = SL_2(\mathbb{C})$, $K = SU(2)$, $AN = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0\right\}$,

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Boundary actions

An action of G on a compact space X is called **strongly proximal** if $\overline{G\nu}$ contains a point mass for any probability measure ν. If the action is in addition minimal, then it is called a **boundary action**.

For any locally compact group G there is a universal boundary action $G \curvearrowright \partial_F G$, $\partial_F G$ is called the **Furstenberg boundary** of G.

Furstenberg's proof of the Poisson formula for $G = KAN$ consists of two major parts:

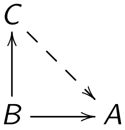
1) $\partial_F G = G/H(G)$, used to prove injectivity of the Poisson integral

$$\mathscr{P}_\nu\colon L^\infty(G/H(G), m) \to H^\infty(G,\mu);$$

2) $H^\infty(G,\mu)^K = \mathbb{C}1$, used to prove surjectivity.

Injective envelopes

A unital G-C*-algebra A is called G-injective if, given unital G-C*-algebras B and C, a completely isometric G-equivariant ucp map $B \to C$ and a G-equivariant ucp map $B \to A$, there is a G-equivariant ucp map $C \to A$ making the diagram



commutative.

Theorem (Hamana, Kalantar-Kennedy)

For any discrete group G, $C(\partial_F G)$ is the injective envelope of $\mathbb{C}$, that is, it is G-injective and every G-equivariant ucp map $C(\partial_F G) \to A$ is completely isometric.

Noncommutative Poisson boundaries

Assume G is a locally compact quantum group, ϕ a normal state on $L^\infty(G)$. Define

$$P_\phi\colon L^\infty(G)\to L^\infty(G),\quad P_\phi(x)=(\phi\otimes\iota)\Delta(x),$$

$$H^\infty(G,\phi)=\{f\in L^\infty(G)\mid P_\phi(x)=x\}.$$

Izumi: the latter is a (right) G-von Neumann algebra with product $x\cdot y=s\text{-}\lim_n P_\phi^n(xy)$.

If $G=\widehat{K}$ for a compact quantum group K, then

$$L^\infty(G)=W^*(K)=\ell^\infty\text{-}\bigoplus_{s\in\operatorname{Irr}(K)}B(H_s).$$

Particularly interested in the (right) $(\operatorname{Ad}K)$-invariant normal states ϕ_μ, where μ is a probability measure on $\operatorname{Irr}(K)$, $W^*(K)^{\operatorname{Ad}K}=\ell^\infty(\operatorname{Irr}(K))$.

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Poisson boundaries of the duals of the q-deformations

Let K be a compact connected semisimple Lie group, $T \subset K$ a maximal torus, K_q the q-deformation of K $(0 < q < 1)$.

The Poisson boundary of $\widehat{K}_q$ for any generating probability measure μ on $\mathrm{Irr}(K)$ is

Izumi: $SU_q(2)/\mathbb{T} \cong S^2_q$ for $K = SU(2)$;

Izumi-N-Tuset: $SU_q(n)/T$ for $K = SU(n)$ $(n \geq 2)$;

Tomatsu: K_q/T for general K.

Remark. For $q = 1$, the Poisson boundary of $\widehat{K}$ is trivial (Biane).

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Drinfeld double

For any compact quantum group K, we can define its **Drinfeld double**

$$D(K) = \text{“}K\hat{K}^{\mathrm{op}}\text{”},\quad L^\infty(D(K)) = L^\infty(K)\bar{\otimes}\ell^\infty(\hat{K}) = L^\infty(K)\bar{\otimes}W^*(K).$$

For compact semisimple Lie groups, $D(K_q)$ is a quantum analogue of $K_{\mathbb{C}}$ (Drinfeld, Pusz-Woronowicz,..., De Commer-Floré, Monk-Voigt).

(*Remark.* For genuine compact groups, $C^*(D(K)) \cong C(K) \rtimes_{\mathrm{Ad}} K$.)

Proposition

For any compact quantum group K and any probability measure μ on $\mathrm{Irr}(K)$, we have a canonical $D(K)$-equivariant isomorphism

$$H^\infty(D(K)^{\mathrm{op}}, h \otimes \phi_\mu) \cong H^\infty(\hat{K}, \mu),$$

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Amenability

For a compact quantum group K, consider a dimension function d on $\operatorname{Rep} K$ ($d \geq 0$, $d \neq 0$):

$$d(U \oplus V) = d(U) + d(V), \qquad d(U \otimes V) = d(U)d(V).$$

For every f.d. representation U consider the matrix

$$\Gamma_U = (\dim \operatorname{Hom}_K(U_s, U \otimes U_t))_{s,t \in \operatorname{Irr}(K)}.$$

The dimension function d is called **amenable** if

$$\|\Gamma_U\|_{\ell^2(\operatorname{Irr}(K))} = d(U) \quad \text{for all} \quad U,$$

equivalently, there are almost $d(U)^{-1}\Gamma_U$-invariant vectors in $\ell^2(\operatorname{Irr}(K))$.

The dimension function d is called **weakly amenable** if there are almost $d(U)^{-1} d\Gamma_U d^{-1}$-invariant nonnegative vectors in $\ell^1(\operatorname{Irr}(K))$.

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There are two natural dimension functions on $\operatorname{Rep} K$ - classical dimension $\dim U$ and quantum dimension $\dim_q U$. They coincide if and only if K is of Kac type ($S^2 = \mathrm{id}$).

The classical dimension function is amenable if and only if there is a state on $\ell^\infty(\widehat{K})$ that is P_ϕ-invariant for all normal states ϕ on $\ell^\infty(\widehat{K}) = W^*(K)$. In this case $\widehat{K}$ is called amenable.

The quantum dimension function is weakly amenable if and only if there is a state on $\ell^\infty(\mathrm{Irr}(K))$ that is P_μ-invariant for all probability measures μ on $\mathrm{Irr}(K)$.

Poisson boundaries and amenability

Theorem (Tomatsu)

Assume K is a compact quantum group with commutative fusion rules, countable $\mathrm{Irr}(K)$ and amenable classical dimension function (so the discrete quantum group $\widehat{K}$ is amenable). Let $H \subset K$ be the largest closed quantum subgroup of Kac type. Then

$$H^\infty(\widehat{K},\mu) \cong L^\infty(K/H)$$

for any generating probability measure μ on $\mathrm{Irr}(K)$.

An important ingredient of the proof is the property

$$H^\infty(\widehat{K},\mu)^K = \mathbb{C}1,$$

which was proved by Hayashi.

Theorem (N-Yamashita, Habbestad-Hataishi-N)

Assume K is a compact quantum group with weakly amenable quantum dimension function. Then there is a noncommutative $D(K)$-space $\partial_\Pi \widehat{K}$ such that

1. *the action of K on $\partial_\Pi \widehat{K}$ is ergodic, $C(\partial_\Pi K)$ is braided-commutative, and the dimension function defined by $C(\partial_\Pi K)$ on $\operatorname{Rep} K$ is amenable;*
2. *$C(\partial_\Pi K)$ is an initial object in the category of $D(K)$-algebras as in (1).*

Furthermore, if m is the unique K-invariant state on $C(\partial_\Pi \widehat{K})$, then $\mathscr{P}_m$ is a complete order isomorphism of $L^\infty(\partial_\Pi \widehat{K}, m)$ onto

$$H^\infty(\widehat{K}) := \{x \in \ell^\infty(\widehat{K}) \mid P_\mu(x) = x \quad \text{for all} \quad \mu\}.$$

Furstenberg-Hamana boundaries of Drinfeld doubles

Theorem

Assume K is a compact quantum group with weakly amenable quantum dimension function. Then $\partial_\Pi \widehat{K}$ is the Furstenberg-Hamana boundary of $D(K)$, that is, $C(\partial_\Pi \widehat{K})$ is the $D(K)$-injective envelope of $\mathbb{C}$.

In particular, if K is a compact connected semisimple Lie group with a fixed maximal torus T, then, for all $0 < q < 1$, the Furstenberg-Hamana boundary of $D(K_q)$ is K_q/T.