Eigenfunctions of the Star graph for all non-zero eigenvalues

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Let $\Gamma = (V, E)$ be a simple graph. We denote by $\Gamma(x)$ the neighbourhood of a vertex $x$. Let $\theta$ be an eigenvalue of the adjacency matrix of $\Gamma$.

**Definition**

A non-zero function $f : V \rightarrow \mathbb{R}$ is called a $\theta$-eigenfunction of $\Gamma$ if for any vertex $x$ of $\Gamma$ the following condition is true

$$\theta \cdot f(x) = \sum_{y \in \Gamma(x)} f(y).$$
Let $G$ be a finite group and $S$ be a generating set of $G$. Let $S$ not contain the identity and be closed with respect to inversion.

**Definition**

The **Cayley graph** $\Gamma = \text{Cay}(G, S)$ of $G$ generated by $S$ is a graph for which:

- the vertex set $V$ is identified with $G$,
- the edge set $E$ is
  \[
  \{ (x, sx) \mid \text{for any } x \in G \text{ and for any } s \in S \}.
  \]

For any vertex $x$ of $\Gamma$ the neighbourhood of $x$ in $\Gamma$ is

\[
\Gamma(x) = Sx = \{ sx \mid s \in S \}.
\]
Let $\Omega$ be the set $\{1, \ldots, n\}$, $n \geq 2$.

We consider the symmetric group $\text{Sym}_\Omega$ and put

$$S = \{ (1\ 2), (1\ 3), \ldots, (1\ n) \}.$$

**Definition**

The **Star graph** $S_n$ is the Cayley graph over the symmetric group $\text{Sym}_\Omega$ with the generating set $S$.
The Star graph is interesting as network topology, because it is an attractive alternative to the hypercube, a popular network for interconnecting processors in a parallel computer, and it compares favorably with hypercube in several aspects.


In this talk, we are primarily interested in the eigenfunctions and the spectrum of the Star graph $S_n$. 
It was shown by Krakovski and Mohar that the spectrum of the Star graph $S_n$ contains all integers from $-n + 1$ to $n - 1$ (except 0 if $n = 2$ or $n = 3$).

Since the Star graph is bipartite, $\mu(n - i) = \mu(-n + i)$ for any integer $i$, where $1 \leq i \leq n$. Moreover, $\pm(n - 1)$ are simple eigenvalues of $S_n$.

Let \( \mathbb{C} \) be a complex field and let \( \mathbb{C}[G] \) be the group algebra of \( G \) over \( \mathbb{C} \).

For a subset \( S \) in \( G \), consider the element \( S \in \mathbb{C}[G] \) given by

\[
S = \sum_{s \in S} s.
\]

Left multiplication of elements from \( \mathbb{C}[G] \) by \( S \) is a linear transformation of \( \mathbb{C}[G] \).

It is known that the matrix of this linear transformation coincides with the adjacency matrix of \( \text{Cay}(G, S) \).

Hence, spectral theory of Cayley graphs is connected with the theory of group representations.
In [CF2012], Chapuy and Feray pointed out that the spectrum of the Star graph $S_n$ can be found using the Jucys theory. In [J1974], Jucys regarded the action of the element equal sum of transpositions

$$J_n = (1 \ n) + (2 \ n) + \ldots + (n-1 \ n)$$

on the complex group algebra over symmetric group $\mathbb{C}[\text{Sym}_n]$ by left multiplication.

We have to swap 1 and $n$, in these transpositions to obtain the generating set of the Star graph $S_n$.


However, eigenfunctions corresponding to the eigenvalues of the Star graph $S_n$ were never obtained explicitly.

In this talk we remind the definition of $PI$-eigenfunctions of the Star graph $S_n$, $n \geq 3$ for any eigenvalue $n - m - 1$, where $n > 2m > 0$ from [1], and we present a generalization of this family for all non-zero eigenvalues from [2].


Let’s define two $m$-tuples:

- $P_m = ((j_1, k_1), (j_2, k_2), \ldots, (j_m, k_m))$, where $2m$ pairwise distinct elements from the set $\{2, \ldots, n\}$ are arranged into $m$ pairs.
- $I_m = (i_1, i_2, \ldots, i_m)$ of $m$ pairwise distinct elements from the set $\{1, \ldots, n\}$;

Let $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ be a permutation from $\text{Sym}_n$.

For any $t$ in $\{2, \ldots, n\}$ one of the following statements hold:

- $\pi_{j_t} = i_t$,
- $\pi_{k_t} = i_t$,
- $\pi_{j_t} \neq i_t$ and $\pi_{k_t} \neq i_t$. 

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Let’s define two $m$-tuples:

- $P_m = ((j_1, k_1), (j_2, k_2), \ldots, (j_m, k_m))$, where $2m$ pairwise distinct elements from the set $\{2, \ldots, n\}$ are arranged into $m$ pairs.

- $I_m = (i_1, i_2, \ldots, i_m)$ of $m$ pairwise distinct elements from the set $\{1, \ldots, n\}$.

Let $\pi = [\pi_1 \pi_2 \ldots \pi_n]$ be a permutation from $\text{Sym}_n$.

For any $t$ in $\{2, \ldots, n\}$ one of the following statements hold:

\[\pi_{j_t} = i_t,\]
\[\pi_{k_t} = i_t,\]
\[\pi_{j_t} \neq i_t \text{ and } \pi_{k_t} \neq i_t.\]
If in $\pi$ for any $t \in \{1, 2, \ldots, m\}$ either $\pi_{jt} = i_t$ or $\pi_{kt} = i_t$ hold, we define a vector $X_\pi = (x_1, x_2, \ldots, x_m)$, where

$$x_t = \begin{cases} 
1, & \text{if } \pi_{jt} = i_t, \\
0, & \text{if } \pi_{kt} = i_t.
\end{cases}$$

**Definition**

We define the function $f = f_{I_m}^{P_m} : \text{Sym}_n \to \mathbb{R}$ as follows.

- If there exists $t$ such that $\pi_{jt} \neq i_t$ and $\pi_{kt} \neq i_t$, then $f = 0$.
- If for any $t$ either $\pi_{jt} = i_t$ or $\pi_{kt} = i_t$, then

$$f = \begin{cases} 
1, & \text{if } wt(X_\pi) \text{ is an even number}, \\
-1, & \text{if } wt(X_\pi) \text{ is an odd number}.
\end{cases}$$
If in $\pi$ for any $t \in \{1, 2, \ldots, m\}$ either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$ hold, we define a vector $X_{\pi} = (x_1, x_2, \ldots, x_m)$, where

$$x_t = \begin{cases} 1, & \text{if } \pi_{j_t} = i_t, \\ 0, & \text{if } \pi_{k_t} = i_t. \end{cases}$$

**Definition**

We define the function $f = f^{P_m}_{I_m} : \text{Sym}_n \to \mathbb{R}$ as follows.

- If there exists $t$ such that $\pi_{j_t} \neq i_t$ and $\pi_{k_t} \neq i_t$, then $f = 0$.
- If for any $t$ either $\pi_{j_t} = i_t$ or $\pi_{k_t} = i_t$, then

$$f = \begin{cases} 1, & \text{if } \text{wt}(X_{\pi}) \text{ is an even number}, \\ -1, & \text{if } \text{wt}(X_{\pi}) \text{ is an odd number}. \end{cases}$$
 PI-eigenfunctions of the Star graph $S_n$

$$ f = f(2, 6, 4) \quad n = 8, \ m = 3 \quad n - m - 1 = 4. $$

$$ \hat{f} = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array} \right) \quad f(\pi) = ? $$

$$ X = (0, 0, 1) \quad f(\pi) = -1 $$

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Eigenfunctions of Star Graph
Theorem

For $n \geq 3$ and $n > 2m > 0$ the function $f_{I_m}^{P_m}$ is an eigenfunction with eigenvalue $n - m - 1$ of the Star graph $S_n$.

The eigenfunctions from this Theorem we call $\Pi$-eigenfunctions.

Obviously, such a function cannot be defined for $m \geq n/2$. 
Further we present a generalization of this family of PI-eigenfunctions of the Star graph $S_n$, $n \geq 3$, for all its non-zero eigenvalues.

To present a generalization of PI-eigenfunctions, we need some new definitions.
Definition of new eigenfunctions

Let \( \Omega \) be the set \( \{ 1, \ldots, n \} \), \( n \geq 3 \).

For any positive integers \( m \) and \( c \) such that \( m < n - 1 \) and \( c < n - m - 1 \), we define a function from \( \text{Sym}_\Omega \) to \( \mathbb{R} \) depending on the choice of two subsets \( \Delta, \Lambda \) of \( \Omega \) and two tuples \( P_\Delta, I_\Lambda \) as follows.

Let \( \Delta \) be an \((m + c)\)-element subset of \( \Omega \setminus \{ 1 \} \), and \( \Delta_1, \ldots, \Delta_c \) be its partition with \( |\Delta_t| = \delta_t \), where \( 1 \leq t \leq c \). For each \( t \), let \( P_t = (j_{t1}, \ldots, j_{t\delta_t}) \) be a tuple consisting of all elements of the set \( \Delta_t \), i.e. all components of \( P_t \) are pairwise distinct. Denote \( P_\Delta = (P_1, \ldots, P_c) \).

Let \( \Lambda \) be an \( m \)-element subset of \( \Omega \), and \( \Lambda_1, \ldots, \Lambda_c \) be its partition with \( |\Lambda_t| = \lambda_t = \delta_t - 1 \), where \( 1 \leq t \leq c \). For each \( t \), let \( I_t = (i_{t1}, \ldots, i_{t\lambda_t}) \) be a tuple consisting of all elements of the set \( \Lambda_t \), i.e. all components of \( I_t \) are pairwise distinct. Denote \( I_\Lambda = (I_1, \ldots, I_c) \).
Definition of new eigenfunctions

Let $\Omega$ be the set $\{1, \ldots, n\}$, $n \geq 3$.

For any positive integers $m$ and $c$ such that $m < n - 1$ and $c < n - m - 1$, we define a function from $\text{Sym}_\Omega$ to $\mathbb{R}$ depending on the choice of two subsets $\Delta, \Lambda$ of $\Omega$ and two tuples $P_\Delta, I_\Lambda$ as follows.

Let $\Delta$ be an $(m + c)$-element subset of $\Omega \setminus \{1\}$, and $\Delta_1, \ldots, \Delta_c$ be its partition with $|\Delta_t| = \delta_t$, where $1 \leq t \leq c$. For each $t$, let $P_t = (j_{t1}, \ldots, j_{t\delta_t})$ be a tuple consisting of all elements of the set $\Delta_t$, i.e. all components of $P_t$ are pairwise distinct. Denote $P_\Delta = (P_1, \ldots, P_c)$.

Let $\Lambda$ be an $m$-element subset of $\Omega$, and $\Lambda_1, \ldots, \Lambda_c$ be its partition with $|\Lambda_t| = \lambda_t = \delta_t - 1$, where $1 \leq t \leq c$. For each $t$, let $I_t = (i_{t1}, \ldots, i_{t\lambda_t})$ be a tuple consisting of all elements of the set $\Lambda_t$, i.e. all components of $I_t$ are pairwise distinct. Denote $I_\Lambda = (I_1, \ldots, I_c)$. 
Definition of new eigenfunctions

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For any positive integers $m$ and $c$ such that $m < n - 1$ and $c < n - m - 1$, we define a function from $\text{Sym}_\Omega$ to $\mathbb{R}$ depending on the choice of two subsets $\Delta$, $\Lambda$ of $\Omega$ and two tuples $P_\Delta$, $I_\Lambda$ as follows.

Let $\Delta$ be an $(m + c)$-element subset of $\Omega \setminus \{1\}$, and $\Delta_1, \ldots, \Delta_c$ be its partition with $|\Delta_t| = \delta_t$, where $1 \leq t \leq c$. For each $t$, let $P_t = (j_{t1}, \ldots, j_{t\delta_t})$ be a tuple consisting of all elements of the set $\Delta_t$, i.e. all components of $P_t$ are pairwise distinct. Denote $P_\Delta = (P_1, \ldots, P_c)$.

Let $\Lambda$ be an $m$-element subset of $\Omega$, and $\Lambda_1, \ldots, \Lambda_c$ be its partition with $|\Lambda_t| = \lambda_t = \delta_t - 1$, where $1 \leq t \leq c$. For each $t$, let $I_t = (i_{t1}, \ldots, i_{t\lambda_t})$ be a tuple consisting of all elements of the set $\Lambda_t$, i.e. all components of $I_t$ are pairwise distinct. Denote $I_\Lambda = (I_1, \ldots, I_c)$.
Definition of new eigenfunctions

Let $H$ be a subgroup of $\text{Sym}_{\Omega}$ such that $H = \{ \tau \in \text{Sym}_{\Omega} | \tau(\Delta_t) = \Delta_t, 1 \leq t \leq c, \text{ and } \tau(i) = i \text{ for any } i \in \Omega \setminus \Delta \}$, i.e. any permutation $\tau$ from $H$ fixes all points from $\Omega \setminus \Delta$, and $\tau(\Delta_t) = \Delta_t$ means that $\tau(j) \in \Delta_t$ for any $j \in \Delta_t$.

If $\pi \in \text{Sym}_{\Omega}$, then $\pi(P_t) = (\pi(j_{t1}), \ldots, \pi(j_{t\delta_t}))$.

For given $\pi \in \text{Sym}_{\Omega}$ and $\tau \in H$, by the precedence relation

$$I_{\Lambda} \prec \pi(\tau(P_{\Delta}))$$

we mean that for any $t$, $1 \leq t \leq c$, and any $i_{ts} \in \Lambda_t$ there exists $j_{ts'} \in \Delta_t$ such that $i_{ts} = \pi(\tau(j_{ts'}))$, moreover $\tau \circ \pi$ preserves the ordering of any $P_t$. In other words, $I_t$ can be obtained from $\pi(\tau(P_t))$ by deleting one component for any $t$. 
Definition of new eigenfunctions

Let $H$ be a subgroup of $\text{Sym}_\Omega$ such that $H = \\
= \{ \tau \in \text{Sym}_\Omega | \tau(\Delta_t) = \Delta_t, 1 \leq t \leq c, \text{ and } \tau(i) = i \text{ for any } i \in \Omega \setminus \Delta \}$,

i.e. any permutation $\tau$ from $H$ fixes all points from $\Omega \setminus \Delta$, and $\tau(\Delta_t) = \Delta_t$ means that $\tau(j) \in \Delta_t$ for any $j \in \Delta_t$.

If $\pi \in \text{Sym}_\Omega$, then $\pi(P_t) = (\pi(j_{t1}), \ldots, \pi(j_{t\delta}))$.

For given $\pi \in \text{Sym}_\Omega$ and $\tau \in H$, by the precedence relation

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we mean that for any $t$, $1 \leq t \leq c$, and any $i_{ts} \in \Lambda_t$ there exists $j_{ts'} \in \Delta_t$ such that $i_{ts} = \pi(\tau(j_{ts'}))$, moreover $\tau \circ \pi$ preserves the ordering of any $P_t$. In other words, $I_t$ can be obtained from $\pi(\tau(P_t))$ by deleting one component for any $t$. 
For a given $\pi \in \text{Sym}_\Omega$ and $\tau \in H$, the precedence relation does not hold for $I_\Lambda$ and $\pi(\tau(P_\Delta))$ if and only if either there exists $i_{ts}$ such that $i_{ts} \neq \pi(\tau(j_{ts'}))$ for some $j_{ts'} \in \Delta_t$ or $\tau \circ \pi$ does not preserve the ordering of some $P_t$. Let $f = f_{I_\Lambda}^P$.

\begin{equation}
\begin{aligned}
f(\pi) = \\
+1, & \quad \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an even permutation } \tau \in H; \\
-1, & \quad \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an odd permutation } \tau \in H; \\
0, & \quad \text{if } I_\Lambda \not\prec \pi(\tau(P_\Delta)) \text{ for any permutation } \tau \in H.
\end{aligned}
\end{equation}
For a given $\pi \in \text{Sym}_\Omega$ and $\tau \in H$, the precedence relation does not hold for $I_\Lambda$ and $\pi(\tau(P_\Delta))$ if and only if either there exists $i_{ts}$ such that $i_{ts} \neq \pi(\tau(j_{ts}'))$ for some $j_{ts}' \in \Delta_t$ or $\tau \circ \pi$ does not preserve the ordering of some $P_t$. Let $f = f_{I_\Lambda}^{P_\Delta}$.

Definition

$$f(\pi) = \begin{cases} +1, & \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an even permutation } \tau \in H; \\ -1, & \text{if } I_\Lambda \prec \pi(\tau(P_\Delta)) \text{ for an odd permutation } \tau \in H; \\ 0, & \text{if } I_\Lambda \not\prec \pi(\tau(P_\Delta)) \text{ for any permutation } \tau \in H. \end{cases}$$

(1)
Example 1. Let $n = 7$ and $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. We put $m = 3$, $c = 2$ and consider $\Delta = \{2, 4\} \cup \{3, 5, 6\}$, $\Lambda = \{1\} \cup \{3, 4\}$. For given tuples $P_\Delta = ((2, 4), (3, 5, 6))$ and $I_\Lambda = ((1), (3, 4))$, we find the values of $f_{P_\Delta}^{I_\Lambda}$ for some permutations $\pi$.

If $\pi = [7, 1, 4, 5, 3, 6, 2]$, then $\pi_2 = 1$, $\pi_4 = 5 \not\in \Lambda_1$, $\pi_3 = 4$, $\pi_5 = 3$, $\pi_6 = 6 \not\in \Lambda_2$. For odd permutation $\tau = (3\ 5)$ the precedence relation $\prec$ holds since $\pi(\tau(3)) = 3$ and $\pi(\tau(5)) = 4$, therefore, $f_{I_\Lambda}^{P_\Delta}(\pi) = -1$.

If $\pi = [1, 5, 4, 7, 3, 6, 2]$, then $\pi_2 = 5 \not\in \Lambda_1$, $\pi_4 = 7 \not\in \Lambda_1$, and hence $f_{I_\Lambda}^{P_\Delta}(\pi) = 0$. 
Example 1. Let \( n = 7 \) and \( \Omega = \{ 1, 2, 3, 4, 5, 6, 7 \} \). We put \( m = 3, \ c = 2 \) and consider \( \Delta = \{ 2, 4 \} \cup \{ 3, 5, 6 \} \), 
\( \Lambda = \{ 1 \} \cup \{ 3, 4 \} \). For given tuples \( P_\Delta = ((2, 4), (3, 5, 6)) \) and 
\( I_\Lambda = ((1), (3, 4)) \), we find the values of \( f_{I_\Lambda}^{P_\Delta} \) for some permutations \( \pi \).

If \( \pi = [7, 1, 4, 5, 3, 6, 2] \), then \( \pi_2 = 1, \ \pi_4 = 5 \notin \Lambda_1, \ \pi_3 = 4, \ \pi_5 = 3, \ \pi_6 = 6 \notin \Lambda_2 \). For odd permutation \( \tau = (3 \ 5) \) the precedence relation \( \prec \) holds since \( \pi(\tau(3)) = 3 \) and \( \pi(\tau(5)) = 4 \), therefore, \( f_{I_\Lambda}^{P_\Delta}(\pi) = -1 \).

If \( \pi = [1, 5, 4, 7, 3, 6, 2] \), then \( \pi_2 = 5 \notin \Lambda_1, \ \pi_4 = 7 \notin \Lambda_1 \), and hence \( f_{I_\Lambda}^{P_\Delta}(\pi) = 0 \).
Example 1. Let $n = 7$ and $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. We put $m = 3$, $c = 2$ and consider $\Delta = \{2, 4\} \cup \{3, 5, 6\}$, $\Lambda = \{1\} \cup \{3, 4\}$. For given tuples $P_{\Delta} = ((2, 4), (3, 5, 6))$ and $I_{\Lambda} = ((1), (3, 4))$, we find the values of $f_{I_{\Lambda}}^{P_{\Delta}}$ for some permutations $\pi$.

If $\pi = [7, 1, 4, 5, 3, 6, 2]$, then $\pi_2 = 1$, $\pi_4 = 5 \notin \Lambda_1$, $\pi_3 = 4$, $\pi_5 = 3$, $\pi_6 = 6 \notin \Lambda_2$. For odd permutation $\tau = (3 \ 5)$ the precedence relation $\prec$ holds since $\pi(\tau(3)) = 3$ and $\pi(\tau(5)) = 4$, therefore, $f_{I_{\Lambda}}^{P_{\Delta}}(\pi) = -1$.

If $\pi = [1, 5, 4, 7, 3, 6, 2]$, then $\pi_2 = 5 \notin \Lambda_1$, $\pi_4 = 7 \notin \Lambda_1$, and hence $f_{I_{\Lambda}}^{P_{\Delta}}(\pi) = 0$. 
Main Theorem

Theorem

For any integers $n$ and $m$, where $n \geq 3$ and $m < n - 1$, the function $f_{\Lambda I}^P$ is an $(n - m - 1)$-eigenfunction of the Star graph $S_n$. 
Example 2. Let $n = 4$, $m = 2$, $c = 1$. For the tuples $P_\Delta = ((2, 3, 4))$ and $I_\Lambda = ((4, 1))$, the function $f^{P_\Delta}_{I_\Lambda}$ is an 1-eigenfunction of the Star graph $S_4$.

Note that there are no $PI$-eigenfunctions for the eigenvalue 1 of the Star graph $S_4$. 
Example

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Eigenfunctions of Star Graph
Remark 1.
Since the Star graph is bipartite, the $(-n + m + 1)$-eigenfunction is known whenever the $(n - m - 1)$-eigenfunction is known.

Remark 2.
If $|\Lambda_t| = 1$, where $1 \leq t \leq c$, then the new eigenfunction is a $PI$-eigenfunction.
Thank you for your attention!