Some new results on independent domination polynomial of a graph

Saeid Alikhani
Joint with Somayeh Jahari

Department of Mathematics
Yazd University

8th European Congress of Mathematics (8ECM), 20-26 June, 2021.

Outline

Some properties of domination polynomial

Definition

Let $G = (V, E)$ be a simple graph. A set $S \subseteq V(G)$ is a dominating set if $N[S] = V$ or equivalently, every vertex in $V(G) \backslash S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G.

Definition and examples

Some properties of domination polynomial

Definition (S. A. & Y.H. Peng, 2008)

The domination polynomial of a simple graph G of order n is the polynomial

$$D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i,$$

where $d(G,i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G.

Definition and examples

Some properties of domination polynomial

Example

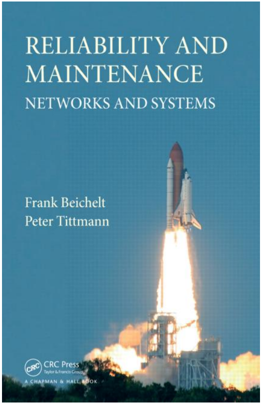
As an example, it is easy to see that for each $n \in \mathbb{N}$,
$D(K_n, x) = (1 + x)^n - 1$.

Example

$D(K_{1,n}, x) = x^n + x(1 + x)^n$.

Applications

Domination reliability is a network reliability measure for some particular kind of service networks, which related to the domination polynomial of a graph.



Some properties of domination polynomial

Definition

The join $G_1 \vee G_2$ of two graph G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2.

Theorem (S. A. & Y.H. Peng, 2008)

Let G_1 and G_2 be graphs of orders n_1 and n_2, respectively. Then

$$D(G_1 \vee G_2, x) =$$

$$\Big((1+x)^{n_1} - 1\Big)\Big((1+x)^{n_2} - 1\Big) + D(G_1, x) + D(G_2, x).$$

Some properties of domination polynomial

Definition

For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H, by adding edges between the i^{th} vertex of G and all vertices of i^{th} copy of H.

Theorem (S. A., 2013)

Let $G = (V, E)$ and $H = (W, F)$ be nonempty graphs of order n and m, respectively. Then

$$D(G \circ H, x) = (x(1 + x)^m + D(H, x))^n.$$

Definition

An independent dominating set of the simple graph $G = (V, E)$ is a vertex subset that is both dominating and independent in G. The *independent domination polynomial* of a graph G is the polynomial

$$D_i(G, x) = \sum_{A} x^{|A|},$$

summed over all independent dominating subsets $A \subseteq V$.

Definition

A root of $D_i(G, x)$ is called an independence domination root.

Theorem (M. Dod, 2016)

If G_1 and G_2 are nonempty graphs, then

$$D_i(G_1 \vee G_2, x) = D_i(G_1, x) + D_i(G_2, x).$$

Theorem (A.E. Brouwer, 2009)

For every graph G, $D(G,1)$ is odd.

Theorem (S. A., 2013)

$D(G,r)$ is odd for every odd integer r. In particular $D(G,1)$ is odd.

Theorem (A.E. Brouwer, 2009)

For every graph G, $D(G,1)$ is odd.

Theorem (S. A., 2013)

$D(G,r)$ is odd for every odd integer r. In particular $D(G,1)$ is odd.

Theorem (S. A., 2013)

Let F be a forest. Then $D(F,-1)=(-1)^{\alpha(F)}$.

Theorem (S. A., 2013)

for each odd number n there is a connected graph G with $D(G,1)=n$.

Theorem (S. Jahari and S. A., 2021)

For any integer number n, there is a connected graph G such that $D_i(G, -1) = n$.

Theorem (S. Jahari and S. A., 2021)

For any negative integer number n, there is a connected graph G for which n is an independence domination root of G.

Theorem (S. Jahari and S. A., 2021)

For any integer number n, there is a connected graph G such that $D_i(G, -1) = n$.

Theorem (S. Jahari and S. A., 2021)

For any negative integer number n, there is a connected graph G for which n is an independence domination root of G.

Definition

For two graphs G and H, let $G[H]$ be the graph with vertex set $V(G) \times V(H)$ and such that vertex (a, x) is adjacent to vertex (b, y) if and only if a is adjacent to b (in G) or $a = b$ and x is adjacent to y (in H). The graph $G[H]$ is the lexicographic product (or composition) of G and H, and can be thought of as the graph arising from G and H by substituting a copy of H for every vertex of G.

Theorem (S. Jahari and S. A., 2021)

If G and H are two graphs, then the independent domination polynomial of $G[H]$ is

$$D_i(G[H], x) = D_i(G, D_i(H, x)).$$

Theorem (S. Jahari and S. A., 2021)

Every graph G is an induced subgraph of a graph H whose independence domination roots lie in $|z| \leq 1$.

Definition

Given two graphs G and H, assume that $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ is a clique cover of G. Construct a new graph from G (which is called compound graph), as follows: for each clique $C_i \in \mathcal{C}$, add a copy of the graph H and join every vertex of C_i to every vertex of H. Let $G^{\Delta}(H)$ denote the new graph. In fact, the compound graph is a generalization of the corona of G and H, if each clique C_i of the clique cover $\mathcal{C}$ is a vertex.

Theorem (S. Jahari and S. A., 2021)

For two graphs G and H, let $\mathcal{C} = \{C_1, C_2, \cdots, C_q\}$ be a clique cover of G. Then

$$D_i(G^{\Delta}(H), x) = D_i^q(H, x) I(G, \frac{x}{D_i(H, x)}).$$

Corollary (S. Jahari and S. A., 2021)

Given two graphs G and H, assume that $\mathcal{C}$ is a clique cover of G. If $|\mathcal{C}| = q$, then $[D_i(H, x)]^{q-\alpha(G)}$ divides $D_i(G^{\Delta}(H), x)$.

Definition

We say that the polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ is unimodal if there exist $k \in \{0, ..., n\}$, called a mode of the sequence such that

$$a_0 \leq a_1 \leq \cdots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \cdots \geq a_n.$$

Definition

A polynomial, $P(x)$, as above is logarithmically concave (or simply log-concave) if for all $k = 1, ..., n-1$, we have

$$a_k^2 \geq a_{k-1} a_{k+1}.$$

Example

An example of a log-concave sequence is the sequence of binomial coefficients $C(n, i)$ for fixed n and $0 \leq i \leq n$.

Example

The sequence of Stirling numbers of the second kind $(S(n, 1), ..., S(n, n))$ is log-concave.

Unimodality problems arise naturally in many branches of mathematics and have been extensively investigated. A basic approach to unimodality problems is to use Newton's inequalities:

Theorem

Let $a_0, a_1, ..., a_n$ be a sequence of nonnegative numbers. Suppose that the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros. Then

$$a_k^2 \geq a_{k-1}a_{k+1}\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right),$$

$$k = 1, 2, ..., n-1,$$

and the sequence is therefore log-concave and unimodal.

Note

$\{constants\ of\ \ real\ rooted\ polynom.\} \subset \{log - concave\} \subset \{unimodal\}$

Note

$A = (1, 7, 10, 3, 2)$ is unimodal but not log-concave ($3^2 \ngeq 2 \times 10$).

Techniques

(i) Real analysis (log-concavity of matching polynomial and the independence polynomial of claw-free graphs),

(ii) Homological algebra (June Huh's proof for the log-concavity of chromatic polynomial)

(iii) Combinatorial arguments (the arguments of Krattenthaler and Hamidoune for proof of log-concavity of matching polynomial and independence polynomial of claw-free graphs.)

Lemma

Let $f(x)$ and $g(x)$ be polynomials with positive coefficients.

(i) If both $f(x)$ and $g(x)$ are log-concave, then so is their product $f(x)g(x)$.

(ii) If $f(x)$ is log-concave, and $g(x)$ is unimodal, then their product $f(x)g(x)$ is unimodal.

(iii) If both $f(x)$ and $g(x)$ are symmetric and unimodal, then so is their product $f(x)g(x)$.

Conjecture (S. A. and Y.H. Peng, 2009)

The domination polynomial of a graph is unimodal.

Theorem (S. A. and Y.H. Peng, 2009)

Let G be a graph of order n. Then for every $0 \leq i < \frac{n}{2}$, we have $d(G,i) \leq d(G,i+1)$.

Corollary (I. Beaton and J.I. Brown, 2021+)

The domination polynomial of P_n and C_n are unimodal.

Theorem (I. Beaton and J.I. Brown, 2021+)

The domination polynomial of complete multipartite graph $K_{n_1,\ldots,n_k}$ is unimodal.

Theorem (I. Beaton and J.I. Brown, 2021+)

Let G be a graph of order n and $k \geq \frac{n}{2}$. If $\frac{d(G,k)}{\binom{n}{k}} \geq \frac{n-k}{k+1}$, then $d(G,i+1) \leq d(G,i)$ for all $i \geq k$.

Corollary (I. Beaton and J.I. Brown, 2021+)

If G is a graph of order n and $\delta(G) \geq 2log_2(n)$, then $D(G,x)$ is unimodal.

Definition

Let $\mathcal{G}(n, p)$ denote the Erdös-Rényi random graph model on n vertices (each edge exists with probability p).

Theorem (I. Beaton and J.I. Brown, 2021+)

Fix $p \in (0, 1)$. Let $G_n \in \mathcal{G}(n, p)$. Then with probability tending to 1, $D(G_n, x)$ is unimodal with mode $\lceil \frac{n}{2} \rceil$.

Theorem (S. Jahari and S. A., 2021)

Given two graphs G and H, let $\mathcal{C}$ be a clique cover of G. Let $D_i(H,x) = ax^2 + bx$, where a, b are nonnegative integers.

(i) If both $I(G,x)$ and $D_i(H,x)$ have only real roots, then so does $D_i(G^{\Delta}(H),x)$.

(ii) If $I(G,x)$ is log-concave and $a = 0$, then so is $D_i(G^{\Delta}(K_b),x)$.

Corollary (S. Jahari and S. A., 2021)

The independent domination polynomial of $K_{t,n} \circ K_1$ is log-concave for every t and is therefore unimodal.

Definition

A (k,n)-path, denoted by P_n^k, begins with k-clique on $\{v_1, v_2, \ldots, v_k\}$. For $i = k+1$ to n, let vertex v_i be adjacent to vertices $\{v_{i-1}, v_{i-2}, \ldots, v_{i-k}\}$ only.

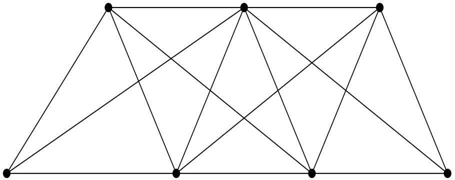


Figure: The 3-path on 7 vertices.

Theorem (S. Jahari and S. A., 2021)

Assume that G is a graph for which $D_i(G,x)$ has only real zeros and $D_i(G,x) = ax^2 + bx$, where $0 < a, b \in \mathbb{N}$. Then $D_i((P_n^k)^\Delta(G),x)$ has only real roots.

Theorem (S. Jahari and S. A., 2021)

Let H be a graph with $\alpha(H) \leq 2$ and $\mathcal{C}$ be a clique cover of another claw-free graph G. If $D_i(H,x)$ has only real zeros, then so does $D_i(G^\Delta(H),x)$. In particular, so does $D_i(G \circ H, x)$.

Example

Consider the centipede graph, $P_n \circ K_1$, and the caterpillar graph, $P_n \circ \bar{K}_2$: Since P_n, i.e., the path with n vertices, is a claw-free graph, so $D_i(P_n \circ K_1, x)$, and $D_i(P_n \circ \bar{K}_2, x)$ have only real roots.

Example

The n-sunlet, $C_n \circ K_1$, where C_n is the cycle with n vertices. So $D_i(C_n \circ K_1, x)$ has only real roots, since C_n is a claw-free graph. In addition, we also can verify that $D_i(C_n \circ K_r, x)$ have only real zeros for $r \geq 1$.

Theorem (S. Jahari and S. A., 2021) (M. Dod, 2016)

For every $n \geq 4$,

$$D_i(P_n, x) = xD_i(P_{n-2}, x) + xD_i(P_{n-3}, x),$$

where $D_i(P_1, x) = x$, $D_i(P_2, x) = 2x$ and $D_i(P_3, x) = x^2 + x$.

Corollary (S. Jahari and S. A., 2021)

For every $n \geq 4$,

$$d_i(P_n, k) = d_i(P_{n-2}, k-1) + d_i(P_{n-3}, k-1)$$

with initial conditions $d_i(P_1, 1) = 1, d_i(P_2, 1) = 2, d_i(P_3, 1) = 1$ and $d_i(P_3, 2) = 1$.

Theorem (S. Jahari and S. A., 2021)

If $F(x, y) = \sum_{n\geq 1}\sum_{k\geq 1} d_i(n, k)x^n y^k$, is the generating function for the number of independent dominating sets of P_n, then

$$F(x, y) = \frac{x(1 + x)^2 y}{1 - (x^2 + x^3)y}$$

Corollary (S. Jahari and S. A., 2021)

For every $k \geq 1$ and $t \geq 0$, the number of independent dominating k-sets of the path P_{k+t} is

$$d_i(P_{k+t}, k) = \binom{k + 1}{t - k + 1}.$$

Definition

Given two integer $n \geq 2$ and $m \geq 4$, a book graph B_n, is defined as follows $V(B_n) = \{u_1, u_2\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(B_n) = \{u_1 u_2\} \cup \{u_1 v_i,\ u_2 w_i,\ v_i w_i : 1 \leq i \leq n\}$. We consider the generalized book graph $B_{n,m}$ with vertex and edge sets by $V(B_{n,m}) = \{u_i : 1 \leq i \leq m-2\} \cup \{v_i, w_i : 1 \leq i \leq n\}$ and $E(B_{n,m}) = \{u_i u_{i+1} : 1 \leq i \leq m-3\} \cup \{u_i w_j : 1 \leq j \leq n,\ i = m-2\} \cup \{u_1 v_i : 1 \leq i \leq n\} \cup \{v_i w_i : 1 \leq i \leq n\}$.

Figure

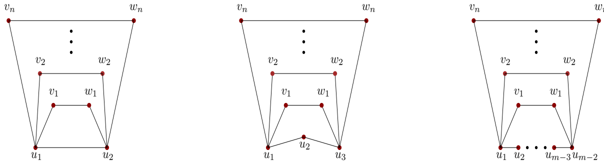


Figure: Graphs B_n, $B_{n,5}$ and $B_{n,m}$, respectively.

Lemma (S. Jahari and S. A., 2021)

The independent domination polynomial of the book graph, B_n, for $n \geq 2$ is given by

$$D_i(B_n, x) = (2^n - 2)x^n + 2x^{n+1}.$$

Corollary (S. Jahari and S. A., 2021)

For each natural number n, $D_i(B_n, x)$ has only real zeros and so is unimodal.

Theorem (S. Jahari and S. A., 2021)

The independent domination polynomial of the generalized book graph, $B_{n,m}$, for $n \geq 2,\ m \geq 3$ is given by

$$(2^n-2)x^n D_i(P_{m-4},x) + 2x^{n+1} D_i(P_{m-5},x) + (x^2+2x^{n+1}) D_i(P_{m-6},x),$$

where for all $j \leq 0, \quad D_i(P_j,x) = 1$.

Theorem (S. Jahari and S. A., 2021)

The independent domination number of the generalized book graph, $B_{n,m}$, for $n \geq 2,\ m \geq 3$ equals to $\gamma_i(B_{n,m}) =$
$\min\{\max\{n, n+\lceil\frac{m-4}{2}\rceil\}, \max\{n+1, n+1+\lceil\frac{m-5}{2}\rceil\}, \max\{2, 2+\lceil\frac{m-6}{2}\rceil\}\}$.

Definition

The friendship (or Dutch-Windmill) graph F_n is a graph that can be constructed by the coalescence of n copies of the cycle graph C_3 of length 3 with a common vertex. Let n and $q \geq 3$ be any positive integer and $F_{q,n}$ be the *generalized friendship graph* formed by a collection of n cycles (all of order q), meeting at a common vertex. The generalized friendship graph may also be referred to as a flower.

Figure

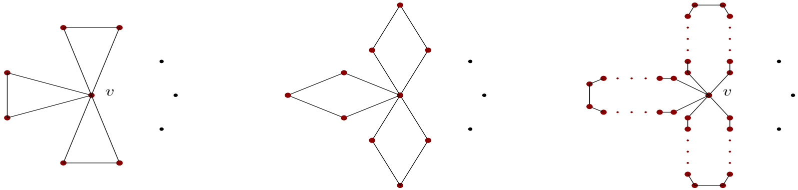


Figure: The graphs F_n, $F_{4,n}$ and $F_{q,n}$, respectively.

Theorem (S. Jahari and S. A., 2021)

(i) The independent domination polynomial of the friendship graph, F_n, for $n \geq 2$ is given by

$$D_i(F_n, x) = x + (2x)^n.$$

(ii) The independent domination polynomial of the generalized friendship graph, $F_{q,n}$, for $n \geq 2$, $q \geq 4$ is given by

$$D_i(F_{q,n}, x) = x(D_i(P_{q-3}, x))^n + nxD_i(P_{q-3}, x)(D_i(P_{q-1}, x))^{n-1}.$$

Theorem (S. Jahari and S. A., 2021)

The independent domination polynomial of friendship graph F_n is unimodal.

Some conjectures and open problems

Conjecture (S. Jahari and S. A., 2021)

The independent domination polynomial of any graph is unimodal.

Open problem

Which graphs have few independence domination roots?

Two graphs G and H are said to be independent dominating equivalent or simply $\mathcal{D}_t$-equivalent, if $D_i(G,x) = D_i(H,x)$. The equivalence class of G, denoted $[G]$, is the set of all graphs $\mathcal{D}_i$-equivalent to G.

Open problem

(i) Which graphs are $\mathcal{D}_i$-unique, that is, are completely determined by their independent domination polynomials?
(ii) Can we determine the $\mathcal{D}_i$-equivalence class of a graph?

Thanks for your attention!