INTEGRATION THE LOADED KORTEWEG-DE VRIES EQUATION IN THE CLASS OF STEPLIKE FUNCTION

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**INTRODUCTION.**

It is known, that the Korteweg-de Vries equation can be integrated with Inverse Scattering Method [1]. In the works [2,3], the Korteweg-de Vries equations with a self-consistent source were integrated for a class of initial data of “step” type; in particular, laws of evolution of the scattering data were established. In applications of the method of inverse scattering transformation one looks for pairs of operators $B$ and $L$ such that the equation has some interesting nonlinear evolution equation for functions $u(x,t)$ that occur as potentials in the operator $L$. For the successful application of the method two further ingredients are needed: 1. the inverse scattering problem must be solved so that the potentials $u(x,t)$ can be reconstructed from scattering data; 2. and that one must be able to determine the evolution of the scattering data with $t$.

In this work, we will consider the loaded Korteweg-de Vries equation

$$u_t - 6uu_x + u_{xxx} + \gamma(t)u(0,t)u_x = 0,$$  \hspace{1cm} (1)

where $u = u(x,t), \ x \in \mathbb{R}, \ t \geq 0, \ \gamma(t) - $ is an arbitrary, continuous function. The function $u = u(x,t)$ is a sufficiently smooth and tending to its limits steplike ($c > 0$)

$$\int_{-\infty}^{0}(1-x)u(x,t)dx + \int_{0}^{\infty}(1+x)u(x,t) - c^2dx + \sum_{k=1}^{3}\int_{-\infty}^{\infty}\frac{\partial^k u(x,t)}{\partial x^k}dx < \infty$$  \hspace{1cm} (2)
The equation (1) is considered with initial condition

$$ u|_{t=0} = u_0(x), \ x \in R^1, \ (3) $$

where $u_0(x)$ function satisfies the conditions ($c > 0$):

1. $\int_{-\infty}^{0}(1-x)|u_0(x)|dx < \infty$, \quad $\int_{0}^{\infty}(1+x)|u_0(x)-c^2|dx < \infty,$

2. Suppose that, the equation $-y'' + u_0(x)y = \lambda y$, \ $x \in R^1$ has $\lambda_1(0), \lambda_2(0), ..., \lambda_N(0)$ negative eigenvalues.

In this work the solution $u(x,t)$ of the loaded Korteweg-de Vries equation (1) in the class of steplike function (2) with initial condition (3) is obtained via Inverse Scattering Method.

**SCATTERING PROBLEM**

Consider the Sturm–Liouville equation on the line \ $x \in R^1$

$$ Ly = -y'' + u(x)y = k^2 y, \quad (4) $$

where the potential \ $u(x)$ is real, locally summable, and has different limits at infinities of different signs:

$$ \lim_{x \to \pm \infty} u(x) = 0, \quad \lim_{x \to \pm \infty} u(x) = c^2, \quad c \geq 0. \quad (5) $$

We assume that \ $u(x)$ tends to its limits fast enough, so that

$$ \int_{-\infty}^{0}(1-x)|u(x)|dx < \infty, \quad \int_{0}^{\infty}(1+x)|u(x)-c^2|dx < \infty. \quad (6) $$

The scattering problem for Eq. (4) under condition (6) was considered in the works [2-4].

We set \ $l = \sqrt{k^2 - c^2}$ \ we choose a branch of the square root in the plane with the cut \ $[-c,c]$ \ such that \ $\text{Im} \ l > 0$ \ for \ $\text{Im} \ k > 0$ and \ $\text{sign} \ l = \text{sign} \ k$ \ for real \ $k$ and \ $|k| > c$, respectively.

Denote by \ $f^+(x,k)$ \ and \ $f^-(x,k)$ \ respectively, the Yost solutions of Eq. (4) with the following asymptotics:
\[ \lim_{x \to \infty} f^+ (x, k) \exp(-ilx) = 1, \quad (\text{Im} \ l \geq 0), \text{and} \quad \lim_{x \to \infty} f^- (x, k) \exp(ikx) = 1, \quad (\text{Im} \ k \geq 0). \quad (7) \]

Under conditions (6), such solutions exist, are unique, and are regular functions of \( l \) and \( k \) for \( \text{Im} \ l > 0 \) and \( \text{Im} \ k > 0 \), respectively. They can be represented in terms of the transformation operators as follows:

\[ f^+ (x, k) = \exp(ilx) + \int_{x}^{\infty} A_1(x, z) \exp(ilz) \, dz, \quad f^- (x, k) = \exp(-ikx) + \int_{-\infty}^{x} A_2(x, z) \exp(-ikz) \, dz. \quad (8) \]

In representation (8), the kernels \( A_1(x, z) \) and \( A_2(x, z) \) are real-valued functions which are related to the potential \( u(x) \) by the equalities

\[ u(x) = 2 \frac{dA_2(x, x)}{dx}, \quad u(x) = c^2 - 2 \frac{dA_1(x, x)}{dx}. \quad (9) \]

Under condition (6), Eq. (4) has solutions \( \psi_1(x, k) \) and \( \psi_2(x, k) \) with the following asymptotics:

\[ \psi_1(x, k) \sim \begin{cases} e^{ikx} + S_{21}(k) e^{-ikx}, & x \to -\infty, \\ S_{22}(k) e^{ikx}, & x \to \infty, \end{cases} \quad (10) \]

\[ \psi_2(x, k) \sim \begin{cases} e^{-ikx} + S_{12}(k) e^{ikx}, & x \to -\infty, \\ S_{11}(k) e^{-ikx}, & x \to \infty \quad (|k| > c). \end{cases} \quad (11) \]

The matrix \( S_{ij}(k), \ i, j = 1, 2 \) is called the \( S \)-matrix of Eq. (4). The coefficients \( S_{ij}(k) \) are continuous in \( k \) \((-\infty < k < \infty)\). The coefficients \( S_{11}(k) \) and \( S_{22}(k) \) are limit values of functions that are meromorphic in the upper half-plane and have simple poles at the points \( k_n = i\chi_n \) \((\chi_n > 0), \ n = 1, 2, 3, \ldots, N, \) where \( \lambda_n = -\chi_n^2 \) are eigenvalues of the operator \( L \) and \( f^-(x, i\chi_n) = B_n f^+(x, i\chi_n). \) Note that \( S_{22}(k) = 0. \)

It is possible to recover the potential \( u(x) \) of Eq. (4) from its \( S \)-matrix. Set

\[ m_n^+ = \left( \int_{-\infty}^{\infty} f^+(x, i\chi_n)^2 \, dx \right)^{-1}, \ n = 1, 2, \ldots, N, \]
The kernel \( A_1(x,y) \) in (8) satisfy the integral Gelfand–Levitan–Marchenko equations:

\[
A_1(x,y) + \Omega_1(x+y) + \oint \Omega_1(z+y)A_1(x,z)dz = 0 \quad (y > x),
\]

where

\[
\Omega_1(x) = \sum_{n=1}^{N} m_n^+ e^{-\sqrt{c^2-x^2}z} + \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{12}(k)e^{ikx}dk + \frac{1}{2\pi} \int_{0}^{c} \left| S_{22}(k) \right|^2 e^{-\sqrt{c^2-k^2}x}dk.
\]

Now the potential \( u(x) \) is determined by formulas (9).

The \( S \)-matrix and the set \( \{ \chi_1, \chi_2, ..., \chi_N, m_1^+, m_2^+, ..., m_N^+ \} \) are called the scattering date of Eq. (4).

Following theorem is valid

**Theorem.** If the potential \( u(x,t) \) is a solution the problem (1)-(3) then the scattering data of the Eq. (4) with the function \( u(x,t) \) depend on \( t \) as follows:

\[
\frac{dS_{21}}{dt} = -(8ik^3 - 2ik\gamma(t)u(0,t))S_{21}, \quad (\text{Im} k = 0) \quad \text{for} \quad |k| > c
\]

\[
\frac{dS_{22}}{dt} = (-4ik^3 + (2c^2 + 4k^2)i)l - i\gamma(t)u(0,t)(l - k))S_{22}, \quad \text{for} \quad |k| \leq c
\]

\[
\frac{dS_{22}}{dt} = \left( -4ik^3 + (2c^2 + 4k^2)\sqrt{c^2-k^2} + \frac{\sqrt{c^2-k^2}}{4\pi} J_1 - \frac{ik}{2} \gamma(t)u(0,t)\left(1 + |S_{21}|^2\right)dx + \frac{\sqrt{c^2-k^2}}{2\pi} \text{Im} J_2 \right) S_{22},
\]

and

\[
\frac{d\chi_n}{dt} = 0, \quad n = 1,2,3, ..., N,
\]

\[
\frac{dB_n}{dt} = \left( -4\chi_n - (2c^2 + 4k^2)\sqrt{c^2-k^2} + \gamma(t)u(0,t)\left(\sqrt{\chi_n^2 + c^2} + \chi_n\right) \right) B_n, \quad n = 1,2,3, ..., N,
\]

where
\[
J_1(t) = \text{v.p.} \int_{-c}^{c} \frac{2k'^2(1+|S_{23}|)^2 \gamma(t)u(0,t)}{k' \sqrt{c^2 - k'^2} (k' - k)} dk',
\]
\[
J_2(t) = \left( \int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) - \frac{2\gamma(t)u(0,t)(l-k')}{(k'^2 - c^2)(k' - k)} dk'.
\]

The taken relations determine completely the evolution of the scattering data for the Eq. (4) which allows as to find the solution of problem for (1-3) by using Inverse scattering problem method.

REFERENCES


