Chernoff approximation of operator semigroups and applications

Yana A. Kinderknecht (Butko)

Technische Universität Braunschweig

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Survey paper

Yana A. Butko (2020)
The method of Chernoff Approximation
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Chernoff approximation of Markov evolution

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- Then

$$(T_t)_{t\geqslant 0} \quad \text{with} \quad T_t f_0(x) := \int f_0(y) P(t,x,dy)$$

  is an operator semigroup (i.e. $T_0 = \mathrm{Id}$, $T_t \circ T_s = T_{t+s}$).

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  is an operator semigroup (i.e. $T_0 = \mathrm{Id}$, $T_t \circ T_s = T_{t+s}$).

- And (if $(T_t)_{t\geqslant 0}$ is C_0 on some BS X)

$$f(t,x) := T_t f_0(x) \equiv \int f_0(y) P(t,x,dy) \equiv \mathbb{E}\left[f_0(\xi_t) \mid \xi_0 = x\right]$$

  solves the Cauchy problem for the following evolution equation:

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) & = & Lf(t,x), \\ f(0,x) & = & f_0(x), \end{cases}$$

  where $T_t = e^{tL}$.

Chernoff approximation of Markov evolution

Stochastics

To determine the transition kernel $P(t, x, dy)$ for a given process $(\xi_t)_{t \geqslant 0}$.

$$\Updownarrow$$

Functional Analysis

To construct the semigroup $T_t \equiv e^{tL}$ with a given generator L.

$$\Updownarrow$$

PDEs

To solve the Cauchy problem for a given evolution equation $\frac{\partial f}{\partial t} = Lf$.

Chernoff approximation of Markov evolution

Example:

- Heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f, \qquad x \in \mathbb{R}^d.$$

- Heat semigroup

$$T_t f_0(x) := (2\pi t)^{-d/2} \int\limits_{\mathbb{R}^d} f_0(y) \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy.$$

- Transition kernel of Brownian motion

$$P(t, x, dy) = (2\pi t)^{-d/2} \exp\left\{-\frac{|x-y|^2}{2t}\right\} dy.$$

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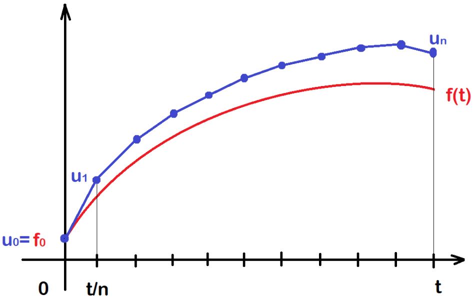
$\Rightarrow$ Numerical schemes for PDEs (time discretizetion):

$$u_0 := f_0, \qquad u_k := F(t/n)u_{k-1}, \quad k = 1, \dots, n, \qquad f(t,\cdot) \approx u_n.$$



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$\Rightarrow$ Numerical schemes for SDEs:

$$(\xi_k^n)_{k=1,\dots,n}, : \qquad \mathbb{E}\left[f_0(\xi_k^n)\,|\,\xi_{k-1}^n\right] = F(t/n)f_0(\xi_{k-1}^n)$$

$$\Rightarrow \qquad \mathbb{E}^x\left[f_0(\xi_t)\right] = \lim_{n\to\infty} \mathbb{E}^x\left[f_0(\xi_n^n)\right]$$

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$\Rightarrow$ deriving / approximation of path integrals (Feynman-Kac formulae).

Chernoff approximation of Markov evolution

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- evolution equation:

$$\frac{\partial f}{\partial t} = \frac{a(x)}{2}\frac{\partial^2 f}{\partial x^2}, \qquad f(0,x) = f_0(x)$$

- Chernoff approximation $F(t)$:

$$F(t)f_0(x) := (2\pi a(x)t)^{-1/2}\int_{\mathbb{R}} f_0(y)\exp\left\{-\frac{|x-y|^2}{2a(x)t}\right\}dy$$

$$= \int_{\mathbb{R}} f_0(y)P(a(x)t, x, dy),$$

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- Feynman formula

$$f(t,x) = T_t f_0(x) = \lim_{n\to\infty}[F(t/n)]^n f_0(x)$$

$$= \lim_{n\to\infty}\int_{\mathbb{R}}\dots\int_{\mathbb{R}} f_0(x_n)P(a(x_{n-1})t/n, x_{n-1}, dx_n)\dots P(a(x)t/n, x, dx_1)$$

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$$= \mathbb{E}^x[f_0(\xi_t)], \qquad \xi_t \quad \text{solves SDE} \quad d\xi_t = \sqrt{a(\xi_t)}dB_t$$

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- Euler scheme for SDE $d\xi_t = \sqrt{a(\xi_t)}dB_t$ on $[0,t]$ with step t/n:

$$\xi_k^n = \xi_{k-1}^n + \sqrt{a(\xi_{k-1}^n)t/n}\, Y_{k-1}, \quad Y_0, \ldots Y_{n-1} \sim \mathcal{N}(0,1), iid$$

$\Rightarrow$

$$\mathbb{E}[f_0(\xi_k^n)|\xi_{k-1}^n] = \mathbb{E}\left[f_0\left(x + \sqrt{a(x)t/n}\, Y_{k-1}\right)\right]\Bigg|_{x=\xi_k^n} = F(t/n)f_0(\xi_k^n)$$

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Hence (tower property)

$$\mathbb{E}[f_0(\xi^n_n)\,|\,\xi^n_0 = x] = \mathbb{E}[\mathbb{E}[f_0(\xi^n_n)\,|\,\xi^n_{n-1}]\,|\,\xi^n_0 = x] = \dots = F^n(t/n)f_0(x).$$

And

$$\mathbb{E}[f_0(\xi_t)\,|\,\xi_0 = x] = T_t f_0(x) = \lim_{n\to\infty} F^n(t/n)f_0(x) = \lim_{n\to\infty}\mathbb{E}[f_0(\xi^n_n)\,|\,\xi^n_0 = x].$$

Chernoff Theorem (1968): Let $(T_t)_{t\geqslant 0}$ be a strongly continuous semigroup on X with generator $(L, \mathrm{Dom}(L))$. Let $(F(t))_{t\geqslant 0}$ be a family of bounded linear operators on X. Assume that

- $F(0) = \mathrm{Id}$,
- $\|F(t)\| \leqslant e^{wt}$ for some $w \in \mathbb{R}$, and all $t \geqslant 0$,
- $\lim\limits_{t\to 0} \dfrac{F(t)\varphi - \varphi}{t} = L\varphi$

  for all $\varphi \in D$, where D is a core for $(L, \mathrm{Dom}(L))$.

Then it holds

$$T_t\varphi = \lim_{n\to\infty} [F(t/n)]^n \varphi, \quad \forall\ \varphi \in X,$$

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- $F(0) = \mathrm{Id}$, *(consistency)*
- $\|F(t)\| \leqslant e^{wt}$ for some $w \in \mathbb{R}$, and all $t \geqslant 0$, *(stability)*
- $\lim\limits_{t\to 0} \dfrac{F(t)\varphi - \varphi}{t} = L\varphi$ *(consistency)*

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Meta-theorem of Numerics: Consistency + stability $\Rightarrow$ convergence.

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$$L \text{ is bdd} \quad \Rightarrow \quad F(t) := \mathrm{Id} + tL \quad \sim \quad e^{tL} \quad \Rightarrow$$

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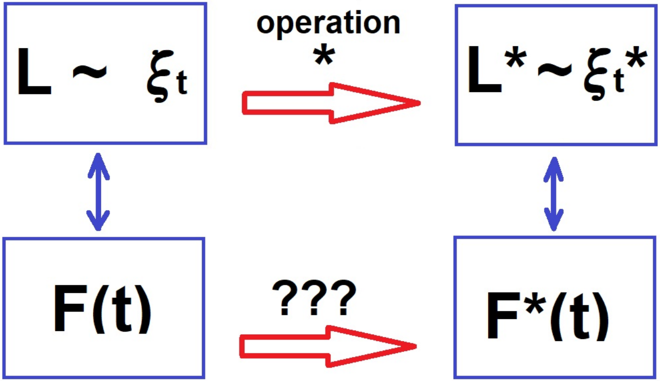

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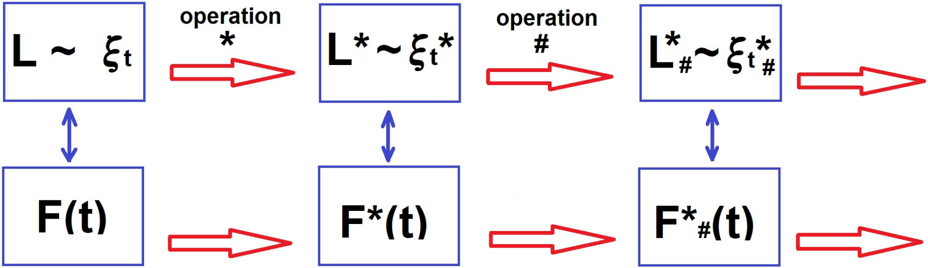

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   - Feller processes in $\mathbb{R}^d$ (Böttcher, Butko, Schilling, Schnurr, Smolyanov 2009-2012)
   - Feller diffusions in a compact Riemannian manifold (Smolyanov, Weizsäcker, Wittich 1999-2007; Mazzucchi, Moretti, Remizov, Smolyanov 2020)

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- operator splitting: $L^* := L_1 + \ldots + L_m$
- averaging of generators: $L^* := \int L_\varepsilon \mu(d\varepsilon)$
- multiplicative perturbations of L $\Leftrightarrow$ random time change of ξ_t via an additive functional: $L^* := aL$
- subordination: $L^* := -f(-L), \quad \xi^*_t := \xi_{\eta_t}$
- "rotation": $L^* := iL$
- killing of ξ_t upon leaving a domain $G \subset \mathbb{R}^d$:
  $L^* := L +$ Dirichlet boundary / external conditions
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$$E^*(D^*) \subset D$$

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2. We have
$$F^*(t) := R_t \circ F(t) \circ E^* \quad \sim \quad e^{tL^*}$$

$R_t : X \to Y$ is a restriction to $\overline{G}$ / multiplication with $\psi_t \to 1_G$

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$$E^*(D^*) \subset D$$
E^* are known for:
   - Dirichlet BC $\quad \varphi = 0 \quad$ on ∂G
   - Robin BC $\quad \frac{\partial \varphi}{\partial n} + \beta\varphi = 0 \quad$ on ∂G, $\quad \beta \geqslant 0$ smooth
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$R_t : X \to Y$ is a restriction to $\overline{G}$ / multiplication with $\psi_t \to 1_G$

Example:

- evolution equation:

$$\begin{aligned} &\frac{\partial f}{\partial t} = \frac{a(x)}{2}\frac{\partial^2 f}{\partial x^2}, && t>0, \quad x\in G\\ &f(0,x) = f_0(x), && x\in G\\ &f(t,x)=0, && t>0, \quad x\in\partial G. \end{aligned}$$

- Chernoff approximation on the base of $F(t)$:

$$F(t)f_0(x) := (2\pi a(x)t)^{-1/2}\int_{\mathbb{R}} f_0(y)\exp\left\{-\frac{|x-y|^2}{2a(x)t}\right\}dy$$

$$= \int_{\mathbb{R}} f_0(y)P(a(x)t,x,dy),$$

- $\Longrightarrow$ Feynman formula

$$f(t,x) =$$

$$= \lim_{n\to\infty}\int_G\dots\int_G f_0(x_n)P(a(x_{n-1})t/n,x_{n-1},dx_n)\dots P(a(x)t/n,x,dx_1)$$

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Corollary (Daletskii–Lie–Trotter formula):

$$e^{tL_1} \circ e^{tL_2} \sim e^{t(L_1+L_2)} \quad \text{i.e.} \quad e^{t(L_1+L_2)} = \lim_{n\to\infty} \left[e^{tL_1/n} \circ e^{tL_2/n} \right]^n$$

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Let $m = 2$. Then $F^*(t) \quad \sim \quad e^{tL^*}$ for

$$F^*(t) := \tau F_1(t) \circ F_2(t) + (1-\tau) F_2(t) \circ F_1(t), \quad \tau \in [0,1]$$

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$\theta = \frac{1}{2} \quad \Rightarrow \quad$ symmetric Strang splitting (scheme of 2nd order)

Averaging

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$\mathcal{E}$ is a parameter set, μ is a probability measure on $\mathcal{E}$

$$L^* := \int_{\mathcal{E}} L_\varepsilon \mu(d\varepsilon)$$

Then

$$F(t) := \int_{\mathcal{E}} e^{tL_\varepsilon} \mu(d\varepsilon) \quad \sim \quad e^{t \int_{\mathcal{E}} L_\varepsilon \mu(d\varepsilon)} \equiv e^{tL^*}$$

Chernoff approximation for Feller semigroups / processes

Feller process $\quad \xi_t \quad \rightsquigarrow T_t \equiv e^{tL} \quad$ on $C_\infty(\mathbb{R}^d) \quad$ with $L \equiv -\widehat{H}$:

$$\widehat{H}\varphi(x) := (2\pi)^{-d} \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} e^{ip\cdot(x-q)} H(x,p)\varphi(q)\,dq\,dp,$$
$$\equiv \left(\mathcal{F}^{-1} \circ H(x,\cdot) \circ \mathcal{F}\varphi\right)(x)$$

where $H(x,\cdot)$ is given by the Lévy–Khintchine formula

$$H(x,p) = C(x) + iB(x)\cdot p + p\cdot A(x)p + \int\limits_{y\neq 0} \left(1 - e^{iy\cdot p} + \frac{iy\cdot p}{1+|y|^2}\right) N(x,dy).$$

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Example: non-degenerate diffusion ($N \equiv 0,\ C \equiv 0$):

$$F(t)\varphi(x) = \frac{1}{\sqrt{(4\pi t)^d \det A(x)}}\int\limits_{\mathbb{R}^d}\varphi(q)e^{-\frac{(x-q-tB(x))\cdot A^{-1}(x)(x-q-tB(x))}{4t}}dq$$

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$(Z_k)_{k=0,\dots,n-1}$ are i.i.d., $\sim N(0, \mathrm{id})$, $X_k \perp Z_k$.

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$$\mathbb{E}[f_0(X_n)\,|\,X_0 = x] = \mathbb{E}[\mathbb{E}[f_0(X_n)\,|\,X_{n-1}]\,|\,X_0 = x] = \ldots = F^n(t/n)f_0(x).$$

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$$\mathbb{E}[f_0(\xi_t) \,|\, \xi_0 = x] = e^{tL}f_0(x) = \lim_{n\to\infty} F^n(t/n)f_0(x) = \lim_{n\to\infty} \mathbb{E}[f_0(X_n) \,|\, X_0 = x].$$

Approximation of non-Markovian evolution

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Some physical, chemical, biological phenomena can be modelled via evolution equations of the form

$$\partial_t^\beta f = Lf, \qquad \beta \in (0,1),$$

where ∂_t^β is the **Caputo derivative** of order β:

$$\partial_t^\beta u(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{u(r)}{(t-r)^\beta} dr - \frac{t^{-\beta}}{\Gamma(1-\beta)} u(0+),$$

and $(L, \mathrm{Dom}(L))$ generates a unif. bdd C_0-SG $(T_t)_{t\geqslant 0}$ (on a BS X).

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More general: We consider equations of the form:

$$\mathcal{D}_t^\mu f(t) = Lf(t),$$

where $\mathcal{D}_t^\mu$ is the **Caputo derivative of distributed order** given by a finite Borel measure μ on $(0,1)$:

$$\mathcal{D}_t^\mu u(t) := \int_0^1 \partial_t^\beta u(t) \mu(d\beta).$$

Approximation of non-Markovian evolution

Consider **"inverse subordinator"** $(E_t^\mu)_{t \geqslant 0}$ (inverse to $(\xi_t^\mu)_{t \geqslant 0}$):

$$E_t^\mu := \inf \left\{ \tau \geqslant 0 \ : \ \xi_\tau^\mu > t \right\},$$

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Then E_t^μ

- has a.s. non-decreasing paths;
- is **non-Markovian!!!**;
- has a smooth PDF $p^\mu(t,x)$.

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Theorem (Hahn, Kobayashi, Umarov, Mijena, Nane, 2012-2014):
Consider the family $(\mathcal{T}_t)_{t\geqslant 0}$ of linear operators on X such that

$$\mathcal{T}_t\varphi := \int_0^\infty T_s\varphi\, p^{\mu}(t,s)ds, \qquad \varphi \in X.$$

Then $(\mathcal{T}_t)_{t\geqslant 0}$ is a strongly continuous family (**not a SG!!!**) and, for $\forall\, f_0 \in \mathrm{Dom}(L)$, the function

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Theorem: Let $F(t) \sim T_t$. Consider $f_n : [0,\infty) \to X$ such that

$$f_n(t) := \int_0^\infty F^n(t/n) f_0\, p^\mu(t,s)ds.$$

Then, for all $\forall\, f_0 \in \mathrm{Dom}(L)$,

$$\|f_n(t) - f(t)\|_X \to 0, \quad n \to \infty,$$

locally uniformly w.r.t. $t \geqslant 0$.

Approximation of non-Markovian evolution

Example: **time-space-fractional diffusion** with $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$, $x \in \mathbb{R}^d$:

$$\partial_t^{1/2} f(t,x) = a(x)\left(-(-\Delta)^{1/2}\right) f(t,x), \qquad f(0,x) = f_0(x).$$

Then

$$p^{1/2}(t,\tau) = \frac{1}{\sqrt{\pi t}} e^{-\frac{\tau^2}{4t}},$$

and $\quad F(t) \sim T_t \equiv e^{t\left[a\left(-(-\Delta)^{1/2}\right)\right]} \quad$ where

$$F(t)\varphi(x) := \Gamma\left(\frac{d+1}{2}\right) \int\limits_{\mathbb{R}^d} \varphi(q) \frac{a(x)t}{\left(\pi|x-q|^2 + a^2(x)t^2\right)^{\frac{d+1}{2}}} dq.$$

Hence for any $x_0 \in \mathbb{R}^d$:

$$f(t,x_0) = \lim_{n\to\infty} \Gamma^n\left(\frac{d+1}{2}\right) \times$$

$$\times \int\limits_0^\infty \int\limits_{\mathbb{R}^{nd}} \left[\prod_{k=1}^n \frac{a(x_{k-1})\tau/n}{\left(\pi|x_k - x_{k-1}|^2 + (a(x_{k-1})\tau/n)^2\right)^{\frac{d+1}{2}}}\right] \frac{f_0(x_n)}{\sqrt{\pi t}} e^{-\frac{\tau^2}{4t}} dx_1 \cdots dx_n d\tau.$$

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