The Slow-Coloring Game

Douglas B. West

Departments of Mathematics
Zhejiang Normal University and
University of Illinois at Urbana-Champaign
dwest@illinois.edu

slides and three papers available on DBW preprint page

Joint work with
Grzegorz Gutowski, Tomasz Krawczyk,
Krzysztof Maziarz, Michał Zając, Xuding Zhu,
Thomas Mahoney, and Gregory J. Puleo

A Coloring Game

Def. proper coloring - adj verts have different colors.
chromatic number $\chi(G)$ - least number of colors used.

A Coloring Game

Def. proper coloring - adj verts have different colors.
chromatic number $\chi(G)$ - least number of colors used.

We want to properly color G, using color i on step i, but vertices are not all available at each step.

A Coloring Game

Def. proper coloring - adj verts have different colors.
chromatic number $\chi(G)$ - least number of colors used.

We want to properly color G, using color i on step i, but vertices are not all available at each step.

We model worst-case interference of nature by a game.

A Coloring Game

Def. proper coloring - adj verts have different colors.
chromatic number $\chi(G)$ - least number of colors used.

We want to properly color G, using color i on step i, but vertices are not all available at each step.

We model worst-case interference of nature by a game.

Def. Slow-coloring Game - On step i,
Lister marks a nonempty set M of uncolored vertices allowed to get color i, scoring $|M|$.

A Coloring Game

Def. proper coloring - adj verts have different colors.
chromatic number $\chi(G)$ - least number of colors used.

We want to properly color G, using color i on step i, but vertices are not all available at each step.

We model worst-case interference of nature by a game.

Def. Slow-coloring Game - On step i,
Lister marks a nonempty set M of uncolored vertices allowed to get color i, scoring $|M|$.
Painter gives color i to an independent subset of M.

A Coloring Game

Def. proper coloring - adj verts have different colors.
chromatic number $\chi(G)$ - least number of colors used.

We want to properly color G, using color i on step i, but vertices are not all available at each step.

We model worst-case interference of nature by a game.

Def. Slow-coloring Game - On step i,
Lister marks a nonempty set M of uncolored vertices allowed to get color i, scoring $|M|$.
Painter gives color i to an independent subset of M.

The game ends when all vertices are colored.
Lister seeks high total score (inefficient, slow coloring).
Painter seeks low total score.

A Coloring Game

Def. proper coloring - adj verts have different colors.
chromatic number $\chi(G)$ - least number of colors used.

We want to properly color G, using color i on step i, but vertices are not all available at each step.

We model worst-case interference of nature by a game.

Def. Slow-coloring Game - On step i,
Lister marks a nonempty set M of uncolored vertices allowed to get color i, scoring $|M|$.
Painter gives color i to an independent subset of M.

The game ends when all vertices are colored.
Lister seeks high total score (inefficient, slow coloring).
Painter seeks low total score.

Def. sum-color cost $\mathring{s}(G)$ - score under optimal play.

Examples

Ex. $\mathring{s}(K_n) = \binom{n+1}{2}$. Lister keeps marking everything.

Examples

Ex. $\mathring{s}(K_n) = \binom{n+1}{2}$. Lister keeps marking everything.

If Lister always marks all uncolored vertices, then a vertex colored i is marked i times.

Examples

Ex. $\mathring{s}(K_n) = \binom{n+1}{2}$. Lister keeps marking everything.

If Lister always marks all uncolored vertices, then a vertex colored i is marked i times.

Painter's best result is then the chromatic sum $\Sigma(G)$ - the least sum of colors on vertices when G is properly colored by positive integers. (Kubicka [1989, 2004]).

Examples

Ex. $\mathring{s}(K_n) = \binom{n+1}{2}$. Lister keeps marking everything.

If Lister always marks all uncolored vertices, then a vertex colored i is marked i times.

Painter's best result is then the chromatic sum $\Sigma(G)$ - the least sum of colors on vertices when G is properly colored by positive integers. (Kubicka [1989, 2004]).

Prop. Always $\mathring{s}(G) \geq \Sigma(G)$.

Examples

Ex. $\mathring{s}(K_n) = \binom{n+1}{2}$. Lister keeps marking everything.

If Lister always marks all uncolored vertices, then a vertex colored i is marked i times.

Painter's best result is then the chromatic sum $\Sigma(G)$ - the least sum of colors on vertices when G is properly colored by positive integers. (Kubicka [1989, 2004]).

Prop. Always $\mathring{s}(G) \geq \Sigma(G)$. Equality for K_n and paths.

Examples

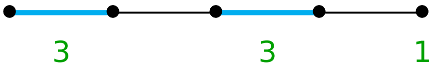
Ex. $\mathring{s}(K_n) = \binom{n+1}{2}$. Lister keeps marking everything.

If Lister always marks all uncolored vertices, then a vertex colored i is marked i times.

Painter's best result is then the chromatic sum $\Sigma(G)$ - the least sum of colors on vertices when G is properly colored by positive integers. (Kubicka [1989, 2004]).

Prop. Always $\mathring{s}(G) \geq \Sigma(G)$. Equality for K_n and paths.

Ex. $\mathring{s}(P_n) = \lfloor 3n/2 \rfloor$.



Lister can get $\lfloor 3n/2 \rfloor$ by scoring 3 per pair.

Examples

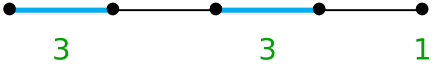
Ex. $\mathring{s}(K_n) = \binom{n+1}{2}$. Lister keeps marking everything.

If Lister always marks all uncolored vertices, then a vertex colored i is marked i times.

Painter's best result is then the chromatic sum $\Sigma(G)$ - the least sum of colors on vertices when G is properly colored by positive integers. (Kubicka [1989, 2004]).

Prop. Always $\mathring{s}(G) \geq \Sigma(G)$. Equality for K_n and paths.

Ex. $\mathring{s}(P_n) = \lfloor 3n/2 \rfloor$.



Lister can get $\lfloor 3n/2 \rfloor$ by scoring 3 per pair.
(Upper bound later.)

Observations (Mahoney–Puleo–West [2018])

The definition of the game yields a recurrence for $\mathring{s}(G)$:

Observations (Mahoney–Puleo–West [2018])

The definition of the game yields a recurrence for $\mathring{s}(G)$:

Obs. Always $\mathring{s}(G) = \max_M (|M| + \min_{I \subseteq M} \mathring{s}(G - I))$.

Observations (Mahoney–Puleo–West [2018])

The definition of the game yields a recurrence for $\mathring{s}(G)$:

Obs. Always $\mathring{s}(G) = \max_M(|M| + \min_{I \subseteq M} \mathring{s}(G - I))$.

Obs. Lower bounds = strategies (algorithms) for Lister.
Upper bounds = strategies (algorithms) for Painter.

Observations (Mahoney–Puleo–West [2018])

The definition of the game yields a recurrence for $\mathring{s}(G)$:

Obs. Always $\mathring{s}(G) = \max_M(|M| + \min_{I \subseteq M} \mathring{s}(G - I))$.

Obs. Lower bounds = strategies (algorithms) for Lister.
Upper bounds = strategies (algorithms) for Painter.

Obs. Always some optimal move for Lister is a set inducing a connected subgraph.

Observations (Mahoney–Puleo–West [2018])

The definition of the game yields a recurrence for $\mathring{s}(G)$:

Obs. Always $\mathring{s}(G) = \max_M(|M| + \min_{I \subseteq M} \mathring{s}(G - I))$.

Obs. Lower bounds = strategies (algorithms) for Lister.
Upper bounds = strategies (algorithms) for Painter.

Obs. Always some optimal move for Lister is a set inducing a connected subgraph.

Obs. Always an optimal response by Painter is a maximal independent set in $G[M]$.

Observations (Mahoney–Puleo–West [2018])

The definition of the game yields a recurrence for $\mathring{s}(G)$:

Obs. Always $\mathring{s}(G) = \max_M(|M| + \min_{I \subseteq M} \mathring{s}(G - I))$.

Obs. Lower bounds = strategies (algorithms) for Lister.
Upper bounds = strategies (algorithms) for Painter.

Obs. Always some optimal move for Lister is a set inducing a connected subgraph.

Obs. Always an optimal response by Painter is a maximal independent set in $G[M]$.

Obs. If $V_1,\ldots,V_k$ partition $V(G)$, then $\mathring{s}(G) \geq \sum \mathring{s}(G[V_i])$.

Observations (Mahoney–Puleo–West [2018])

The definition of the game yields a recurrence for $\mathring{s}(G)$:

Obs. Always $\mathring{s}(G) = \max_M(|M| + \min_{I\subseteq M} \mathring{s}(G-I))$.

Obs. Lower bounds = strategies (algorithms) for Lister.
Upper bounds = strategies (algorithms) for Painter.

Obs. Always some optimal move for Lister is a set inducing a connected subgraph.

Obs. Always an optimal response by Painter is a maximal independent set in $G[M]$.

Obs. If $V_1,\ldots,V_k$ partition $V(G)$, then $\mathring{s}(G) \geq \sum \mathring{s}(G[V_i])$.

(This gave the lower bound for paths.)

Stars

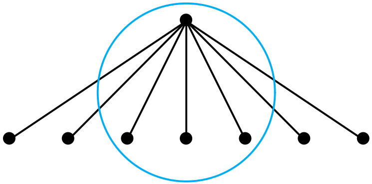
Thm. $\mathring{s}(K_{1,r}) = r + 1 + u_r$,

where $u_r = \max\{k : \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

Stars

Thm. $\mathring{s}(K_{1,r}) = r + 1 + u_r$,
where $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

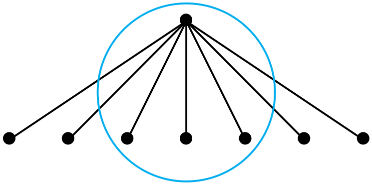
Pf. Lister plays the center and some p leaves.



Stars

Thm. $\mathring{s}(K_{1,r}) = r + 1 + u_r$,
where $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

Pf. Lister plays the center and some p leaves.

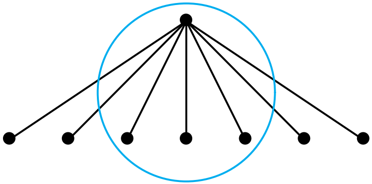


$p < u_r \Rightarrow$ Painter plays center, result is $r + 1 + p$.

Stars

Thm. $\mathring{s}(K_{1,r}) = r + 1 + u_r$,
where $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

Pf. Lister plays the center and some p leaves.



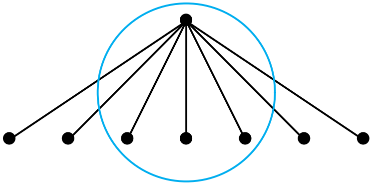
$p < u_r \Rightarrow$ Painter plays center, result is $r + 1 + p$.

$p > u_r \Rightarrow$ Painter plays leaves. Inductively, score
$p + 1 + \mathring{s}(K_{1,r-p}) = r + 1 + 1 + u_{r-p} \le r + 1 + u_r$.

Stars

Thm. $\mathring{s}(K_{1,r}) = r + 1 + u_r$,
where $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

Pf. Lister plays the center and some p leaves.



$p < u_r \Rightarrow$ Painter plays center, result is $r + 1 + p$.

$p > u_r \Rightarrow$ Painter plays leaves. Inductively, score
$$p + 1 + \mathring{s}(K_{1,r-p}) = r + 1 + 1 + u_{r-p} \le r + 1 + u_r.$$

$p = u_r \Rightarrow$ either Painter move yields score $r + 1 + u_r$. ■

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \leq \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.

Always score $\le \chi(G) \cdot$ #colored. ∎

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G) n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.
Always score $\le \chi(G) \cdot$ #colored. ∎

Thm. (Wu) $\mathring{s}(G) \ge n + \sum_{i<j} u_{r_i} u_{r_j}$ when G is a complete k-partite n-vertex graph with part-sizes $r_1, \ldots, r_k$.

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.
Always score $\le \chi(G) \cdot$ #colored. ∎

Thm. (Wu) $\mathring{s}(G) \ge n + \sum_{i<j} u_{r_i} u_{r_j}$ when G is a complete k-partite n-vertex graph with part-sizes $r_1, \ldots, r_k$.

- If all $r_i = \frac{n}{k}$, then $u_{r_i} \sim \sqrt{\frac{2n}{k}}$, so $\mathring{s}(G) \sim n + \binom{k}{2}\frac{2n}{k} = kn$.

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.

Always score $\le \chi(G) \cdot$ #colored. ■

Thm. (Wu) $\mathring{s}(G) \ge n + \sum_{i<j} u_{r_i} u_{r_j}$ when G is a complete k-partite n-vertex graph with part-sizes $r_1, \ldots, r_k$.

- If all $r_i = \frac{n}{k}$, then $u_{r_i} \sim \sqrt{\frac{2n}{k}}$, so $\mathring{s}(G) \sim n + \binom{k}{2}\frac{2n}{k} = kn$.

Pf. Lister marks u_{r_i} verts. in the ith part, scoring $\sum_i u_{r_i}$.

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.

Always score $\le \chi(G) \cdot$ #colored. ■

Thm. (Wu) $\mathring{s}(G) \ge n + \sum_{i<j} u_{r_i} u_{r_j}$ when G is a complete k-partite n-vertex graph with part-sizes $r_1, \ldots, r_k$.

- If all $r_i = \frac{n}{k}$, then $u_{r_i} \sim \sqrt{\frac{2n}{k}}$, so $\mathring{s}(G) \sim n + \binom{k}{2}\frac{2n}{k} = kn$.

Pf. Lister marks u_{r_i} verts. in the ith part, scoring $\sum_i u_{r_i}$.

Painter colors in some part t, leaving $r_t - u_{r_t}$ there.

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.

Always score $\le \chi(G) \cdot$ #colored. ■

Thm. (Wu) $\mathring{s}(G) \ge n + \sum_{i<j} u_{r_i} u_{r_j}$ when G is a complete k-partite n-vertex graph with part-sizes $r_1, \ldots, r_k$.

• If all $r_i = \frac{n}{k}$, then $u_{r_i} \sim \sqrt{\frac{2n}{k}}$, so $\mathring{s}(G) \sim n + \binom{k}{2}\frac{2n}{k} = kn$.

Pf. Lister marks u_{r_i} verts. in the ith part, scoring $\sum_i u_{r_i}$.

Painter colors in some part t, leaving $r_t - u_{r_t}$ there.

Fixing t, apply the induction hypoth and $u_{r-u_r} \ge u_r - 1$:

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.

Always score $\le \chi(G) \cdot$ #colored. ■

Thm. (Wu) $\mathring{s}(G) \ge n + \sum_{i<j} u_{r_i}u_{r_j}$ when G is a complete k-partite n-vertex graph with part-sizes $r_1, \ldots, r_k$.

- If all $r_i = \frac{n}{k}$, then $u_{r_i} \sim \sqrt{\frac{2n}{k}}$, so $\mathring{s}(G) \sim n + \binom{k}{2}\frac{2n}{k} = kn$.

Pf. Lister marks u_{r_i} verts. in the ith part, scoring $\sum_i u_{r_i}$.

Painter colors in some part t, leaving $r_t - u_{r_t}$ there.

Fixing t, apply the induction hypoth and $u_{r-u_r} \ge u_r - 1$:

$\mathring{s}(G) \ge \sum_i u_{r_i} + n - u_{r_t} + \sum_{i<j} u_{r_i}u_{r_j} - \sum_{i \ne t} u_{r_i}(u_{r_t} - u_{r_t - u_{r_t}})$ ■

A Sharp Upper Bound

Prop. If G has n vertices, then $\mathring{s}(G) \le \chi(G)n$.

Pf. If Lister marks m verts, Painter colors at least $\frac{m}{\chi(G)}$.
Always score $\le \chi(G) \cdot$ #colored. ■

Thm. (Wu) $\mathring{s}(G) \ge n + \sum_{i<j} u_{r_i} u_{r_j}$ when G is a complete k-partite n-vertex graph with part-sizes $r_1, \ldots, r_k$.

• If all $r_i = \frac{n}{k}$, then $u_{r_i} \sim \sqrt{\frac{2n}{k}}$, so $\mathring{s}(G) \sim n + \binom{k}{2} \frac{2n}{k} = kn$.

Pf. Lister marks u_{r_i} verts. in the ith part, scoring $\sum_i u_{r_i}$.
Painter colors in some part t, leaving $r_t - u_{r_t}$ there.
Fixing t, apply the induction hypoth and $u_{r-u_r} \ge u_r - 1$:
$\mathring{s}(G) \ge \sum_i u_{r_i} + n - u_{r_t} + \sum_{i<j} u_{r_i} u_{r_j} - \sum_{i \ne t} u_{r_i}(u_{r_t} - u_{r_t - u_{r_t}})$ ■

Ques. How to improve the upper bound $\chi(G)n$ for special classes of graphs?

Results I - Trees

Thm. (Mahoney–Puleo–West [2018]) If G is n-vertex tree, then $\mathring{s}(G) \le \left\lfloor \frac{3}{2}n \right\rfloor$, equality for paths (stars minimize).

Results I - Trees

Thm. (Mahoney–Puleo–West [2018]) If G is n-vertex tree, then $\mathring{s}(G) \le \left\lfloor \frac{3}{2}n \right\rfloor$, equality for paths (stars minimize).

Thm. (Puleo–West [2019]) Linear-time algorithm for trees, inductive formula, and extremal n-vertex trees.

Results I - Trees

Thm. (Mahoney–Puleo–West [2018]) If G is n-vertex tree, then $\mathring{s}(G) \le \left\lfloor \frac{3}{2}n \right\rfloor$, equality for paths (stars minimize).

Thm. (Puleo–West [2019]) Linear-time algorithm for trees, inductive formula, and extremal n-vertex trees.

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$, with equality if and only if T contains a spanning forest in which every vertex has degree 1 or 3,

Results I - Trees

Thm. (Mahoney–Puleo–West [2018]) If G is n-vertex tree, then $\mathring{s}(G) \le \left\lfloor \frac{3}{2} n \right\rfloor$, equality for paths (stars minimize).

Thm. (Puleo–West [2019]) Linear-time algorithm for trees, inductive formula, and extremal n-vertex trees.

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$, with equality if and only if T contains a spanning forest in which every vertex has degree 1 or 3, except for one vertex of degree 0 or 6 when n is odd.

Results I - Trees

Thm. (Mahoney–Puleo–West [2018]) If G is n-vertex tree, then $\mathring{s}(G) \le \left\lfloor \frac{3}{2} n \right\rfloor$, equality for paths (stars minimize).

Thm. (Puleo–West [2019]) Linear-time algorithm for trees, inductive formula, and extremal n-vertex trees.

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$, with equality if and only if T contains a spanning forest in which every vertex has degree 1 or 3, except for one vertex of degree 0 or 6 when n is odd.

Thm. Among n-vertex trees, the star $K_{1,n-1}$ is the unique minimizer of $\mathring{s}(T)$ if neither $n-1$ nor $n-2$ is a triangular number (having the form $\binom{k+1}{2}$). Otherwise, also $\mathring{s}(T) = n + u_{n-1}$ also for a few others.

Results I - Trees

Thm. (Mahoney–Puleo–West [2018]) If G is n-vertex tree, then $\mathring{s}(G) \le \left\lfloor \frac{3}{2}n \right\rfloor$, equality for paths (stars minimize).

Thm. (Puleo–West [2019]) Linear-time algorithm for trees, inductive formula, and extremal n-vertex trees.

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$, with equality if and only if T contains a spanning forest in which every vertex has degree 1 or 3, except for one vertex of degree 0 or 6 when n is odd.

Thm. Among n-vertex trees, the star $K_{1,n-1}$ is the unique minimizer of $\mathring{s}(T)$ if neither $n-1$ nor $n-2$ is a triangular number (having the form $\binom{k+1}{2}$). Otherwise, also $\mathring{s}(T) = n + u_{n-1}$ also for a few others.

Thm. On forests, $\mathring{s}$ equals the "interactive sum choice number" of Bonamy–Meeks [2021], which is the outcome of a game played by Requester and Supplier.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \leq d$ for all $H \subseteq G$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1 + \frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1 + \frac{3}{4}d)n$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$. Worst $2n$?

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1 + \frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1 + \frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$. Worst $2n$?

For planar graphs, 4-colorable yields $\mathring{s}(G) \le 4n$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$. Worst $2n$?

For planar graphs, 4-colorable yields $\mathring{s}(G) \le 4n$.

Thm. When G is acyclically 5-colorable, $\mathring{s}(G) \le 3.986n$.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$. Worst $2n$?

For planar graphs, 4-colorable yields $\mathring{s}(G) \le 4n$.

Thm. When G is acyclically 5-colorable, $\mathring{s}(G) \le 3.986n$.

Thm. If G is 4-colorable, then $\mathring{s}(G) \le \frac{8n+3|E(G)|}{5}$,
so $\mathring{s}(G) < 3.4n$ when G is planar.

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.
Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$. Worst $2n$?

For planar graphs, 4-colorable yields $\mathring{s}(G) \le 4n$.

Thm. When G is acyclically 5-colorable, $\mathring{s}(G) \le 3.986n$.

Thm. If G is 4-colorable, then $\mathring{s}(G) \le \frac{8n+3|E(G)|}{5}$,
so $\mathring{s}(G) < 3.4n$ when G is planar. Worst $\frac{5}{2}n$?

Results II - Degeneracy, etc.

(Gutowski–Krawczyk–Maziarz–West–Zając–Zhu [2021?])

Def. G is d-degenerate if $\delta(H) \le d$ for all $H \subseteq G$.

d-degenerate is $(d+1)$-colorable, so $\mathring{s}(G) \le (1+d)n$.

Copies of K_{d+1} yield $\mathring{s}(G) = (1+\frac{1}{2}d)n$.

Thm. If G is d-degenerate, then $\mathring{s}(G) \le (1+\frac{3}{4}d)n$.

For 2-degenerate graphs, this yields $\mathring{s}(G) \le 2.5n$.

Thm. If G is outerplanar, then $\mathring{s}(G) \le \frac{7}{3}n$.   Worst $2n$?

For planar graphs, 4-colorable yields $\mathring{s}(G) \le 4n$.

Thm. When G is acyclically 5-colorable, $\mathring{s}(G) \le 3.986n$.

Thm. If G is 4-colorable, then $\mathring{s}(G) \le \frac{8n+3|E(G)|}{5}$,
so $\mathring{s}(G) < 3.4n$ when G is planar.   Worst $\frac{5}{2}n$?

Thm. For planar G with Hamiltonian dual, $\mathring{s}(G) \le 3n$.

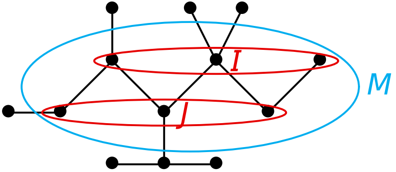
Trees - Upper Bound

Thm. $\mathring{s}(T) \leq \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

Trees - Upper Bound

Thm. $\mathring{s}(T) \leq \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

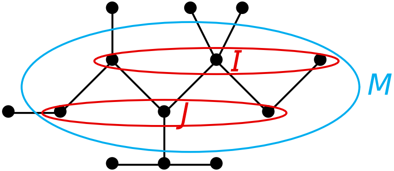
Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter will delete I or J.



Trees - Upper Bound

Thm. $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter will delete I or J.
Note $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\mathring{s}(T - I) + \mathring{s}(T - J)\right]$.



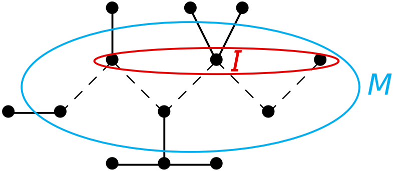
Trees - Upper Bound

Thm. $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter will delete I or J.

Note $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\, \mathring{s}(T - I) + \mathring{s}(T - J)\right]$.

Inductively, $\mathring{s}(T - I) \le \frac{3}{2}(n - |I|) - \frac{1}{2}o(T - I)$.

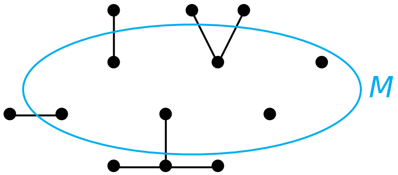


Trees - Upper Bound

Thm. $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter will delete I or J.
Note $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\, \mathring{s}(T-I) + \mathring{s}(T-J) \right]$.
Inductively, $\mathring{s}(T-I) \le \frac{3}{2}(n-|I|) - \frac{1}{2}o(T-I)$.

Let $T' = T - E(T[M])$; one vert. of M in each component.



Trees - Upper Bound

Thm. $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

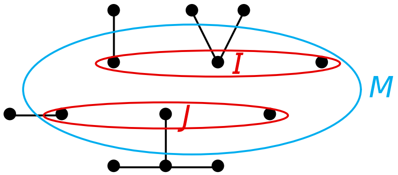
Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I,J) of M; Painter will delete I or J.

Note $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\,\mathring{s}(T-I) + \mathring{s}(T-J)\right]$.

Inductively, $\mathring{s}(T-I) \le \frac{3}{2}(n-|I|) - \frac{1}{2}o(T-I)$.

Let $T' = T - E(T[M])$; one vert. of M in each component.

Whether the component of T' containing $v \in M$ is even or odd, it saves $\frac{1}{2}$ when Painter colors one of I and J.



Trees - Upper Bound

Thm. $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$ for every n-vertex tree T.

Pf. Lister optimally marks M, with $T[M]$ connected.
Given the bipartition (I, J) of M; Painter will delete I or J.

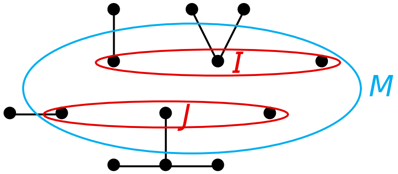
Note $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\mathring{s}(T-I) + \mathring{s}(T-J)\right]$.

Inductively, $\mathring{s}(T-I) \le \frac{3}{2}(n-|I|) - \frac{1}{2}o(T-I)$.

Let $T' = T - E(T[M])$; one vert. of M in each component.

Whether the component of T' containing $v \in M$ is even or odd, it saves $\frac{1}{2}$ when Painter colors one of I and J.

Thus $\mathring{s}(T) \le |M| + \frac{1}{2}\left[\frac{3}{2}(n-|I|) + \frac{3}{2}(n-|J|) - \frac{|M|}{2}\right] = \frac{1}{2}3n$. ■



Inductive Bounds

Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \leq \mathring{s}(G) \leq \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Inductive Bounds

Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \le \mathring{s}(G) \le \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Pf. **Lower:** Lister plays optimally on $G[A]$ and $G[B]$.

Inductive Bounds

Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \le \mathring{s}(G) \le \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Pf. **Lower:** Lister plays optimally on $G[A]$ and $G[B]$.

Upper: Painter wants to respond optimally on $G[A]$ and $G[B]$ separately, to $M \cap A$ and $M \cap B$.

Inductive Bounds

Lem. When G is a graph and $\{A,B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \le \mathring{s}(G) \le \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Pf. **Lower:** Lister plays optimally on $G[A]$ and $G[B]$.

Upper: Painter wants to respond optimally on $G[A]$ and $G[B]$ separately, to $M \cap A$ and $M \cap B$.

This may color both ends of edges in $[A,B]$.

Inductive Bounds

Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \le \mathring{s}(G) \le \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Pf. Lower: Lister plays optimally on $G[A]$ and $G[B]$.

Upper: Painter wants to respond optimally on $G[A]$ and $G[B]$ separately, to $M \cap A$ and $M \cap B$.

This may color both ends of edges in $[A, B]$.

Let S be their ends in B. Painter ignores S and answers $M - S$ in B. Such moves cost at most $\mathring{s}(G[B])$.

Inductive Bounds

Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \le \mathring{s}(G) \le \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Pf. **Lower:** Lister plays optimally on $G[A]$ and $G[B]$.

Upper: Painter wants to respond optimally on $G[A]$ and $G[B]$ separately, to $M \cap A$ and $M \cap B$.

This may color both ends of edges in $[A, B]$.

Let S be their ends in B. Painter ignores S and answers $M - S$ in B. Such moves cost at most $\mathring{s}(G[B])$.

Each edge of $[A, B]$ causes extra cost at most once, since the end in A is colored on that round. ∎

Inductive Bounds

Lem. When G is a graph and $\{A, B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \le \mathring{s}(G) \le \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Pf. **Lower:** Lister plays optimally on $G[A]$ and $G[B]$.

Upper: Painter wants to respond optimally on $G[A]$ and $G[B]$ separately, to $M \cap A$ and $M \cap B$.

This may color both ends of edges in $[A, B]$.

Let S be their ends in B. Painter ignores S and answers $M - S$ in B. Such moves cost at most $\mathring{s}(G[B])$.

Each edge of $[A, B]$ causes extra cost at most once, since the end in A is colored on that round. ■

Idea: For trees, $|[A, B]| = 1$. Find a suitable cut-edge to avoid paying the extra 1.

Inductive Bounds

Lem. When G is a graph and $\{A,B\}$ partitions $V(G)$,
$\mathring{s}(G[A]) + \mathring{s}(G[B]) \le \mathring{s}(G) \le \mathring{s}(G[A]) + \mathring{s}(G[B]) + |[A,B]|$.

Pf. Lower: Lister plays optimally on $G[A]$ and $G[B]$.

Upper: Painter wants to respond optimally on $G[A]$ and $G[B]$ separately, to $M \cap A$ and $M \cap B$.

This may color both ends of edges in $[A,B]$.

Let S be their ends in B. Painter ignores S and answers $M - S$ in B. Such moves cost at most $\mathring{s}(G[B])$.

Each edge of $[A,B]$ causes extra cost at most once, since the end in A is colored on that round. ■

Idea: For trees, $|[A,B]| = 1$. Find a suitable cut-edge to avoid paying the extra 1.

Def. A stem vertex has one non-leaf neighbor (deleting an incident cut-edge yields a star).

Computation for Forests

Def. Let $u_r = \max\{k : \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

Prop. $\mathring{s}(K_{1,n-1}) = n + u_{n-1}$.

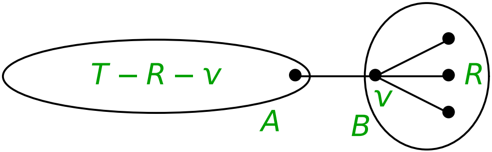
Computation for Forests

Def. Let $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

Prop. $\mathring{s}(K_{1,n-1}) = n + u_{n-1}$.

Thm. Let T be a forest. If v is a stem vertex of T and R is its set of leaf neighbors, with $r = |R|$, then

$$\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v) + r + 1 + u_r, & \text{if } r+1 \text{ is not triangular,} \\ \mathring{s}(T-R) + r + u_r, & \text{if } r+1 \text{ is triangular.} \end{cases}$$



$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r + 1 + u_r$

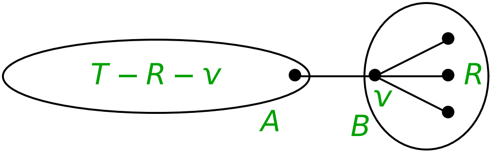
Computation for Forests

Def. Let $u_r = \max\{k\colon \binom{k+1}{2} \le r\} = \left\lfloor \frac{-1+\sqrt{1+8r}}{2} \right\rfloor \approx \sqrt{2r}$.

Prop. $\mathring{s}(K_{1,n-1}) = n + u_{n-1}$.

Thm. Let T be a forest. If v is a stem vertex of T and R is its set of leaf neighbors, with $r = |R|$, then

$$\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v) + r + 1 + u_r, & \text{if } r+1 \text{ is not triangular,} \\ \mathring{s}(T-R) + r + u_r, & \text{if } r+1 \text{ is triangular.} \end{cases}$$



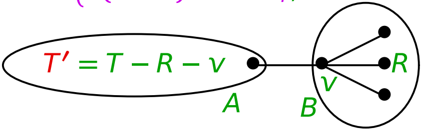
$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r + 1 + u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le \lfloor 3n/2 \rfloor$, with equality if and only if T contains a spanning forest in which every vertex has degree 1 or 3, except for one vertex of degree 0 or 6 when n is odd.

The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$

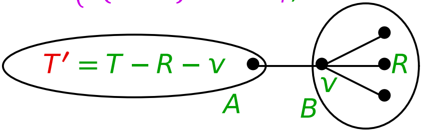


$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r+1+u_r$

The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v) + r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R) + r + u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



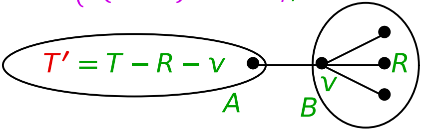
$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r + 1 + u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



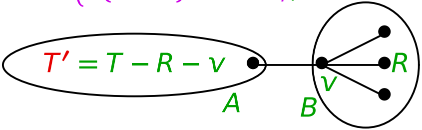
$T[B] = K_{1,r}$
$\mathring{s}(K_{1,r}) = r+1+u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

Pf. Consider the degree $(r+1)$ of a stem vertex v.

The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$

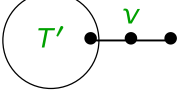


$T[B] = K_{1,r}$
$\mathring{s}(K_{1,r}) = r+1+u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

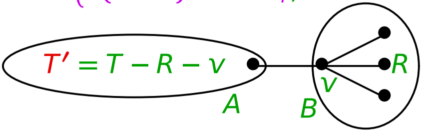
Pf. Consider the degree ($r+1$) of a stem vertex v.

$r=1$ ($r+1$ not triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T') + 3$.



The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$

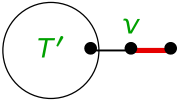


$T[B] = K_{1,r}$
$\mathring{s}(K_{1,r}) = r+1+u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

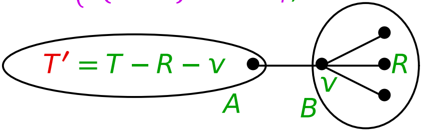
Pf. Consider the degree $(r+1)$ of a stem vertex v.

$r = 1$ ($r+1$ not triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T') + 3$.



The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



$T[B] = K_{1,r}$

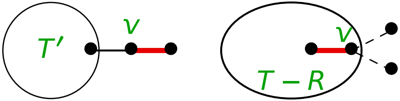
$\mathring{s}(K_{1,r}) = r+1+u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

Pf. Consider the degree $(r+1)$ of a stem vertex v.

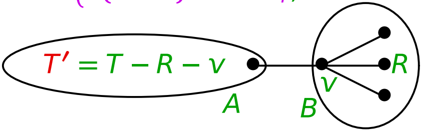
$r=1$ ($r+1$ not triangular), $u_r=1$. $\mathring{s}(T)=\mathring{s}(T')+3$.

$r=2$ ($r+1$ triangular), $u_r=1$. $\mathring{s}(T)=\mathring{s}(T-R)+3$, leaf v.



The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



$T[B] = K_{1,r}$

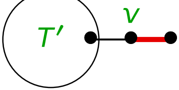
$\mathring{s}(K_{1,r}) = r+1+u_r$

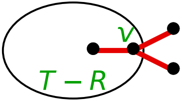
Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

Pf. Consider the degree ($r+1$) of a stem vertex v.

$r=1$ ($r+1$ not triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T') + 3$.

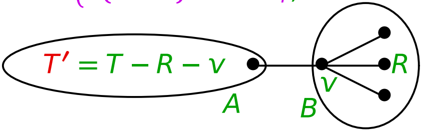
$r=2$ ($r+1$ triangular), $u_r=1$. $\mathring{s}(T) = \mathring{s}(T-R)+3$, leaf v.





The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r+1+u_r$

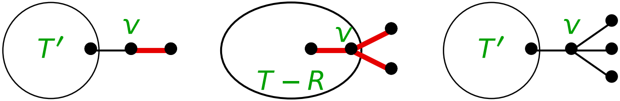
Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

Pf. Consider the degree $(r+1)$ of a stem vertex v.

$r=1$ ($r+1$ not triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T') + 3$.

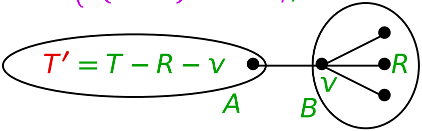
$r=2$ ($r+1$ triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T-R) + 3$, leaf v.

$r=3$ ($r+1$ not triangular), $u_r = 2$. $\mathring{s}(T) = \mathring{s}(T') + 6$.



The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r+1+u_r$

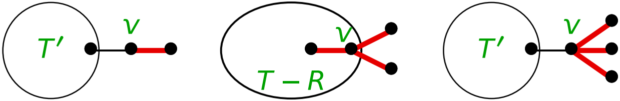
Thm. If T is an n-vertex forest, then $\mathring{s}(T) \leq 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

Pf. Consider the degree $(r+1)$ of a stem vertex v.

$r=1$ ($r+1$ not triangular), $u_r=1$. $\mathring{s}(T) = \mathring{s}(T')+3$.

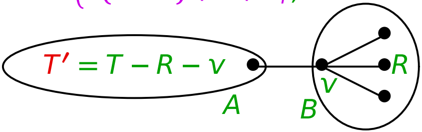
$r=2$ ($r+1$ triangular), $u_r=1$. $\mathring{s}(T) = \mathring{s}(T-R)+3$, leaf v.

$r=3$ ($r+1$ not triangular), $u_r=2$. $\mathring{s}(T) = \mathring{s}(T')+6$.



The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v)+r+1+u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R)+r+u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r + 1 + u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

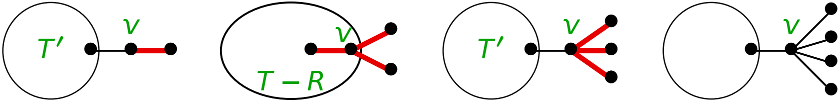
Pf. Consider the degree $(r+1)$ of a stem vertex v.

$r = 1$ ($r+1$ not triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T') + 3$.

$r = 2$ ($r+1$ triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T-R) + 3$, leaf v.

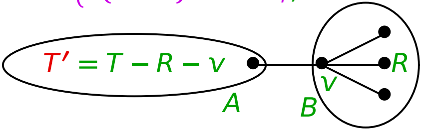
$r = 3$ ($r+1$ not triangular), $u_r = 2$. $\mathring{s}(T) = \mathring{s}(T') + 6$.

$r \ge 4$. In either case, $\mathring{s}(T) < 3n/2$, and no $1,3$-forest.



The Extremal Forests

Thm. $\mathring{s}(T) = \begin{cases} \mathring{s}(T-R-v) + r + 1 + u_r, & \text{if } r+1 \text{ not triangular,} \\ \mathring{s}(T-R) + r + u_r, & \text{if } r+1 \text{ triangular.} \end{cases}$



$T[B] = K_{1,r}$

$\mathring{s}(K_{1,r}) = r + 1 + u_r$

Thm. If T is an n-vertex forest, then $\mathring{s}(T) \le 3n/2$; equality if and only if T has a spanning forest with all degrees 1 or 3.

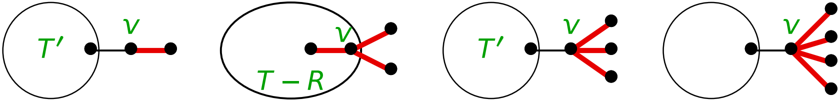
Pf. Consider the degree $(r+1)$ of a stem vertex v.

$r = 1$ ($r+1$ not triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T') + 3$.

$r = 2$ ($r+1$ triangular), $u_r = 1$. $\mathring{s}(T) = \mathring{s}(T-R) + 3$, leaf v.

$r = 3$ ($r+1$ not triangular), $u_r = 2$. $\mathring{s}(T) = \mathring{s}(T') + 6$.

$r \ge 4$. In either case, $\mathring{s}(T) < 3n/2$, and no $1,3$-forest.



The Decomposition Method

Lem. If $V_1, \ldots, V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

The Decomposition Method

Lem. If $V_1, \dots, V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

Pf. Let $w_i = \frac{\sqrt{c_i|V_i|}}{\sum_j \sqrt{c_j|V_j|}}$; note $\sum_i w_i = 1$.

The Decomposition Method

Lem. If $V_1, \ldots, V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

Pf. Let $w_i = \frac{\sqrt{c_i|V_i|}}{\sum_j \sqrt{c_j|V_j|}}$; note $\sum_i w_i = 1$.

When Lister marks M, for some j we have $\frac{|M \cap V_j|}{|M|} \ge w_j$.

The Decomposition Method

Lem. If $V_1, \ldots, V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

Pf. Let $w_i = \frac{\sqrt{c_i|V_i|}}{\sum_j \sqrt{c_j|V_j|}}$; note $\sum_i w_i = 1$.

When Lister marks M, for some j we have $\frac{|M \cap V_j|}{|M|} \ge w_j$.

Painter plays optimally on $G[V_j]$, ignoring the rest of M.

The Decomposition Method

Lem. If $V_1, \ldots, V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

Pf. Let $w_i = \frac{\sqrt{c_i|V_i|}}{\sum_j \sqrt{c_j|V_j|}}$; note $\sum_i w_i = 1$.

When Lister marks M, for some j we have $\frac{|M \cap V_j|}{|M|} \ge w_j$.

Painter plays optimally on $G[V_j]$, ignoring the rest of M.

Note $|M| \le \left|M \cap V_j\right| / w_j$.

The Decomposition Method

Lem. If $V_1, \ldots, V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

Pf. Let $w_i = \frac{\sqrt{c_i|V_i|}}{\sum_j \sqrt{c_j|V_j|}}$; note $\sum_i w_i = 1$.

When Lister marks M, for some j we have $\frac{|M\cap V_j|}{|M|} \ge w_j$.

Painter plays optimally on $G[V_j]$, ignoring the rest of M.
Note $|M| \le \left|M \cap V_j\right| / w_j$.

Rounds played in $G[V_i]$ form a game on $G[V_i]$ played optimally by Painter, so on those rounds $|M \cap V_i|$ totals at most $c_i\,|V_i|$ and $\sum |M|$ in those rounds is at most $\frac{c_i|V_i|}{w_i}$.

The Decomposition Method

Lem. If $V_1, \dots, V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

Pf. Let $w_i = \frac{\sqrt{c_i|V_i|}}{\sum_j \sqrt{c_j|V_j|}}$; note $\sum_i w_i = 1$.

When Lister marks M, for some j we have $\frac{|M \cap V_j|}{|M|} \ge w_j$.

Painter plays optimally on $G[V_j]$, ignoring the rest of M.
Note $|M| \le \left|M \cap V_j\right| / w_j$.

Rounds played in $G[V_i]$ form a game on $G[V_i]$ played optimally by Painter, so on those rounds $|M \cap V_i|$ totals at most $c_i\,|V_i|$ and $\sum |M|$ in those rounds is at most $\frac{c_i|V_i|}{w_i}$.

Over all rounds, $\mathring{s}(G) \le \sum_i \frac{c_i|V_i|}{w_i} = (\sum_i \sqrt{c_i\,|V_i|})^2$. ■

The Decomposition Method

Lem. If $V_1,\ldots,V_t$ partition $V(G)$ so that $\mathring{s}(G[V_i]) \le c_i\,|V_i|$, then $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2$.

Pf. Let $w_i = \frac{\sqrt{c_i|V_i|}}{\sum_j \sqrt{c_j|V_j|}}$; note $\sum_i w_i = 1$.

When Lister marks M, for some j we have $\frac{|M\cap V_j|}{|M|} \ge w_j$.

Painter plays optimally on $G[V_j]$, ignoring the rest of M. Note $|M| \le \left|M\cap V_j\right|/w_j$.

Rounds played in $G[V_i]$ form a game on $G[V_i]$ played optimally by Painter, so on those rounds $|M\cap V_i|$ totals at most $c_i\,|V_i|$ and $\sum |M|$ in those rounds is at most $\frac{c_i|V_i|}{w_i}$.

Over all rounds, $\mathring{s}(G) \le \sum_i \frac{c_i|V_i|}{w_i} = (\sum_i \sqrt{c_i\,|V_i|})^2$. ∎

Cor. $\mathring{s}(K_{r_1,\ldots,r_k}) \le (\sum_i \sqrt{r_i})^2 = n + \sum_{i<j} \sqrt{2r_i}\sqrt{2r_j}$.

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \leq \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2$

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \leq \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\, c_B|B|}$.

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \leq \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\, c_B|B|}$.

Cor. $\mathring{s}(G) \leq (c_A + c_B)\,|V(G)|$.

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \leq \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\,c_B|B|}$.

Cor. $\mathring{s}(G) \leq (c_A + c_B)\,|V(G)|$.

Pf. AM-GM: $2\sqrt{c_A\,|B|}\sqrt{c_B\,|A|} \leq c_A\,|B| + c_B\,|A|$. ■

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \le \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\,c_B|B|}$.

Cor. $\mathring{s}(G) \le (c_A + c_B)\,|V(G)|$.

Pf. AM-GM: $2\sqrt{c_A\,|B|}\sqrt{c_B\,|A|} \le c_A\,|B| + c_B\,|A|$. ■

Cor. $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2 \le \left(\sum_i c_i\right)|V(G)|$. (Induction)

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \leq \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\, c_B|B|}$.

Cor. $\mathring{s}(G) \leq (c_A + c_B)\,|V(G)|$.

Pf. AM-GM: $2\sqrt{c_A\,|B|}\sqrt{c_B\,|A|} \leq c_A\,|B| + c_B\,|A|$. ■

Cor. $\mathring{s}(G) \leq \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2 \leq \left(\sum_i c_i\right)|V(G)|$. (Induction)

Lem. If G is d-degenerate and $\sum_{i=1}^{t}(d_i+1) \geq d+1$, then $V(G)$ splits into $V_1, \ldots, V_t$ so $G[V_i]$ is d_i-degenerate.

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \le \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\,c_B|B|}$.

Cor. $\mathring{s}(G) \le (c_A + c_B)\,|V(G)|$.

Pf. AM-GM: $2\sqrt{c_A\,|B|}\sqrt{c_B\,|A|} \le c_A\,|B| + c_B\,|A|$. ■

Cor. $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2 \le \left(\sum_i c_i\right)|V(G)|$. (Induction)

Lem. If G is d-degenerate and $\sum_{i=1}^{t}(d_i+1) \ge d+1$, then $V(G)$ splits into $V_1, \ldots, V_t$ so $G[V_i]$ is d_i-degenerate.

Pf. Since v_j has at most d earlier neighbors, v_j cannot have $d_i + 1$ earlier neighbors in V_i for each i. ■

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \le \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\,c_B|B|}$.

Cor. $\mathring{s}(G) \le (c_A + c_B)\,|V(G)|$.

Pf. AM-GM: $2\sqrt{c_A\,|B|}\sqrt{c_B\,|A|} \le c_A\,|B| + c_B\,|A|$. ■

Cor. $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2 \le \left(\sum_i c_i\right)|V(G)|$. (Induction)

Lem. If G is d-degenerate and $\sum_{i=1}^{t}(d_i+1) \ge d+1$, then $V(G)$ splits into $V_1, \ldots, V_t$ so $G[V_i]$ is d_i-degenerate.

Pf. Since v_j has at most d earlier neighbors, v_j cannot have $d_i + 1$ earlier neighbors in V_i for each i. ■

Thm. If G is an n-vertex d-degenerate graph, then $\mathring{s}(G) \le (\frac{3}{4}d + 1)n$, and $\mathring{s}(G) \le (\frac{3}{4}d + \frac{3}{4})n$ for odd d.

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \le (\sqrt{c_A|A|} + \sqrt{c_B|B|})^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\, c_B|B|}$.

Cor. $\mathring{s}(G) \le (c_A + c_B)\,|V(G)|$.

Pf. AM-GM: $2\sqrt{c_A\,|B|}\sqrt{c_B\,|A|} \le c_A\,|B| + c_B\,|A|$. ■

Cor. $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2 \le \left(\sum_i c_i\right)|V(G)|$. (Induction)

Lem. If G is d-degenerate and $\sum_{i=1}^{t}(d_i+1) \ge d+1$, then $V(G)$ splits into $V_1, \ldots, V_t$ so $G[V_i]$ is d_i-degenerate.

Pf. Since v_j has at most d earlier neighbors, v_j cannot have $d_i + 1$ earlier neighbors in V_i for each i. ■

Thm. If G is an n-vertex d-degenerate graph, then $\mathring{s}(G) \le (\frac{3}{4}d + 1)n$, and $\mathring{s}(G) \le (\frac{3}{4}d + \frac{3}{4})n$ for odd d.

Pf. Split $V(G)$ into $\lceil d/2 \rceil$ sets inducing 1-degenerate subgraphs, plus one independent set if d is even.

Application to d-Degenerate Graphs

Lem. $\mathring{s}(G) \le \left(\sqrt{c_A|A|} + \sqrt{c_B|B|}\right)^2 = c_A|A| + c_B|B| + 2\sqrt{c_A|A|\,c_B|B|}$.

Cor. $\mathring{s}(G) \le (c_A + c_B)\,|V(G)|$.

Pf. AM-GM: $2\sqrt{c_A\,|B|}\sqrt{c_B\,|A|} \le c_A\,|B| + c_B\,|A|$. ■

Cor. $\mathring{s}(G) \le \left(\sum_i \sqrt{c_i\,|V_i|}\right)^2 \le \left(\sum_i c_i\right)|V(G)|$. (Induction)

Lem. If G is d-degenerate and $\sum_{i=1}^{t}(d_i+1) \ge d+1$, then $V(G)$ splits into $V_1, \ldots, V_t$ so $G[V_i]$ is d_i-degenerate.

Pf. Since v_j has at most d earlier neighbors, v_j cannot have $d_i + 1$ earlier neighbors in V_i for each i. ■

Thm. If G is an n-vertex d-degenerate graph, then $\mathring{s}(G) \le (\frac{3}{4}d + 1)n$, and $\mathring{s}(G) \le (\frac{3}{4}d + \frac{3}{4})n$ for odd d.

Pf. Split $V(G)$ into $\lceil d/2 \rceil$ sets inducing 1-degenerate subgraphs, plus one independent set if d is even.

$\sum c_i = \frac{d}{2}\frac{3}{2} + 1$ for even, $\frac{d+1}{2}\frac{3}{2}$ for odd. ■

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \Rightarrow \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \Rightarrow \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

From $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$, a consequence is $\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \;\Rightarrow\; \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

From $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$, a consequence is $\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.

Useful when we know $|A| \ge \gamma n$.

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \Rightarrow \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

From $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$, a consequence is $\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.

Useful when we know $|A| \ge \gamma n$.

Given an acyclic r-coloring with r odd, let

$A = r-1$ biggest classes, $c_A = \frac{3}{2}\frac{r-1}{2}$;

$B = 1$ smallest class, $c_B = 1$; $\gamma = \frac{r-1}{r}$

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \Rightarrow \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

From $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$, a consequence is $\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.

Useful when we know $|A| \ge \gamma n$.

Given an acyclic r-coloring with r odd, let

$A = r-1$ biggest classes, $c_A = \frac{3}{2}\frac{r-1}{2}$;

$B = 1$ smallest class, $c_B = 1$; $\gamma = \frac{r-1}{r} \ge \frac{3r-3}{3r+1}$.

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \Rightarrow \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

From $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$, a consequence is
$\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.
Useful when we know $|A| \ge \gamma n$.

Given an acyclic r-coloring with r odd, let
$A = r-1$ biggest classes, $c_A = \frac{3}{2}\frac{r-1}{2}$;
$B = 1$ smallest class, $c_B = 1$; $\gamma = \frac{r-1}{r} \ge \frac{3r-3}{3r+1}$.

Thm. $\chi_a(G) \le 3 \Rightarrow \mathring{s}(G) \le \frac{4+2\sqrt{3}}{3} n < 2.4881$

$\chi_a(G) \le 5 \Rightarrow \mathring{s}(G) \le \frac{13+4\sqrt{3}}{5} n < 3.9857$

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \;\Rightarrow\; \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

From $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$, a consequence is $\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.
Useful when we know $|A| \ge \gamma n$.

Given an acyclic r-coloring with r odd, let
$A = r-1$ biggest classes, $c_A = \frac{3}{2}\frac{r-1}{2}$;
$B = 1$ smallest class, $c_B = 1$; $\gamma = \frac{r-1}{r} \ge \frac{3r-3}{3r+1}$.

Thm. $\chi_a(G) \le 3 \;\Rightarrow\; \mathring{s}(G) \le \frac{4+2\sqrt{3}}{3}n < 2.4881$

$\chi_a(G) \le 5 \;\Rightarrow\; \mathring{s}(G) \le \frac{13+4\sqrt{3}}{5}n < 3.9857 \quad < 4$.

Acyclically 5-Colorable Graphs

Def. Acyclic coloring = proper coloring with no 2-colored cycle. $\chi_a(G)$ = fewest colors needed.

Every acyclically r-colorable graph is r-colorable, so $\chi_a(G) \le r \Rightarrow \mathring{s}(G) \le c_r\,|V(G)|$ for some c_r.

From $\mathring{s}(G) \le \left(\sqrt{c_A\,|A|} + \sqrt{c_B\,|B|}\right)^2$, a consequence is $\mathring{s}(G) \le \left(\sqrt{c_A\gamma} + \sqrt{c_B(1-\gamma)}\right)^2 n$ when $\frac{c_A}{c_A+c_B} \le \gamma \le \frac{|A|}{|A|+|B|}$.
Useful when we know $|A| \ge \gamma n$.

Given an acyclic r-coloring with r odd, let
$A = r-1$ biggest classes, $c_A = \frac{3}{2}\frac{r-1}{2}$;
$B = 1$ smallest class, $c_B = 1$; $\gamma = \frac{r-1}{r} \ge \frac{3r-3}{3r+1}$.

Thm. $\chi_a(G) \le 3 \Rightarrow \mathring{s}(G) \le \frac{4+2\sqrt{3}}{3}n < 2.4881$

$\chi_a(G) \le 5 \Rightarrow \mathring{s}(G) \le \frac{13+4\sqrt{3}}{5}n < 3.9857 \quad < 4$.

Thm. (Borodin [1979]) $\chi_a(G) \le 5$ when G is planar.

Approach for Sparse 4-Colorable Graphs

For a graph G, define a potential function by
$\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$,

Approach for Sparse 4-Colorable Graphs

For a graph G, define a potential function by
$\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$,

where $\phi_G(x) = \begin{cases} \frac{3}{5} & \text{if } x \in E(G), \\ 1 + \frac{\min\{d_G(x), 3\}}{5} & \text{if } x \in V(G). \end{cases}$

Approach for Sparse 4-Colorable Graphs

For a graph G, define a potential function by
$\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$,

where $\phi_G(x) = \begin{cases} \frac{3}{5} & \text{if } x \in E(G), \\ 1 + \frac{\min\{d_G(x),3\}}{5} & \text{if } x \in V(G). \end{cases}$

Always $\phi_G(v) \leq \frac{8}{5}$, so $\mathring{s}(G) \leq \Phi(G) \;\Rightarrow\; \mathring{s}(G) \leq \frac{8n+3|E(G)|}{5}$.

Approach for Sparse 4-Colorable Graphs

For a graph G, define a potential function by
$\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$,

where $\phi_G(x) = \begin{cases} \frac{3}{5} & \text{if } x \in E(G), \\ 1 + \frac{\min\{d_G(x), 3\}}{5} & \text{if } x \in V(G). \end{cases}$

Always $\phi_G(v) \leq \frac{8}{5}$, so $\mathring{s}(G) \leq \Phi(G) \Rightarrow \mathring{s}(G) \leq \frac{8n+3|E(G)|}{5}$.

$|E(G)| \leq 3n - 6$ when G is planar, so $\mathring{s}(G) \leq \frac{17}{5}n = 3.4n$.

Approach for Sparse 4-Colorable Graphs

For a graph G, define a potential function by
$\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$,

where $\phi_G(x) = \begin{cases} \frac{3}{5} & \text{if } x \in E(G), \\ 1 + \frac{\min\{d_G(x),3\}}{5} & \text{if } x \in V(G). \end{cases}$

Always $\phi_G(v) \le \frac{8}{5}$, so $\mathring{s}(G) \le \Phi(G) \;\Rightarrow\; \mathring{s}(G) \le \frac{8n+3|E(G)|}{5}$.

$|E(G)| \le 3n-6$ when G is planar, so $\mathring{s}(G) \le \frac{17}{5}n = 3.4n$.

Idea: For any marked set M in a 4-colorable graph G, Painter seeks an independent set $X \subseteq M$ such that $|M| \le \Phi(G) - \Phi(G-X)$; inductively this suffices.

Approach for Sparse 4-Colorable Graphs

For a graph G, define a potential function by
$\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$,

where $\phi_G(x) = \begin{cases} \frac{3}{5} & \text{if } x \in E(G), \\ 1 + \frac{\min\{d_G(x),3\}}{5} & \text{if } x \in V(G). \end{cases}$

Always $\phi_G(v) \le \frac{8}{5}$, so $\mathring{s}(G) \le \Phi(G) \;\Rightarrow\; \mathring{s}(G) \le \frac{8n+3|E(G)|}{5}$.

$|E(G)| \le 3n - 6$ when G is planar, so $\mathring{s}(G) \le \frac{17}{5}n = 3.4n$.

Idea: For any marked set M in a 4-colorable graph G, Painter seeks an independent set $X \subseteq M$ such that $|M| \le \Phi(G) - \Phi(G - X)$; inductively this suffices.

- Proof that such X exists - 4 pages.

Approach for Sparse 4-Colorable Graphs

For a graph G, define a potential function by
$\Phi(G) = \sum_{x \in V(G) \cup E(G)} \phi_G(x)$,

where $\phi_G(x) = \begin{cases} \frac{3}{5} & \text{if } x \in E(G), \\ 1 + \frac{\min\{d_G(x),3\}}{5} & \text{if } x \in V(G). \end{cases}$

Always $\phi_G(v) \le \frac{8}{5}$, so $\mathring{s}(G) \le \Phi(G) \;\Rightarrow\; \mathring{s}(G) \le \frac{8n+3|E(G)|}{5}$.

$|E(G)| \le 3n-6$ when G is planar, so $\mathring{s}(G) \le \frac{17}{5}n = 3.4n$.

Idea: For any marked set M in a 4-colorable graph G, Painter seeks an independent set $X \subseteq M$ such that $|M| \le \Phi(G) - \Phi(G-X)$; inductively this suffices.

- Proof that such X exists - 4 pages.

Comment: Upper bound $3n$ for planar graphs with Hamiltonian dual graph is easy, because the vertices split into two sets inducing trees. $(c_A + c_B = 3)$

Sketch for Outerplanar Graphs

For an induced subgraph H of a maximal outerplanar graph, define potential $\Phi(H) = \sum_{x \in V(H) \cup E(H)} \phi(x, H)$,

Sketch for Outerplanar Graphs

For an induced subgraph H of a maximal outerplanar graph, define potential $\Phi(H) = \sum_{x \in V(H) \cup E(H)} \phi(x, H)$,

where $\phi(x, H) = \begin{cases} 1/3 & \text{if } x \in E(H), \\ 5/3 & \text{if } x \text{ is a vertex in a triangle, else} \\ 5/3 & \text{if } d_H(x) \geq 3, \\ 4/3 & \text{if } d_H(x) \in \{1, 2\}, \\ 1 & \text{if } d_H(x) = 0. \end{cases}$

Sketch for Outerplanar Graphs

For an induced subgraph H of a maximal outerplanar graph, define potential $\Phi(H) = \sum_{x \in V(H) \cup E(H)} \phi(x, H)$,

$$\text{where } \phi(x,H) = \begin{cases} 1/3 & \text{if } x \in E(H), \\ 5/3 & \text{if } x \text{ is a vertex in a triangle, else} \\ 5/3 & \text{if } d_H(x) \geq 3, \\ 4/3 & \text{if } d_H(x) \in \{1,2\}, \\ 1 & \text{if } d_H(x) = 0. \end{cases}$$

Ex. Isolated vertex gives 1, edge 3, triangle 6.

Sketch for Outerplanar Graphs

For an induced subgraph H of a maximal outerplanar graph, define potential $\Phi(H) = \sum_{x \in V(H) \cup E(H)} \phi(x, H)$,

where $\phi(x,H) = \begin{cases} 1/3 & \text{if } x \in E(H), \\ 5/3 & \text{if } x \text{ is a vertex in a triangle, else} \\ 5/3 & \text{if } d_H(x) \geq 3, \\ 4/3 & \text{if } d_H(x) \in \{1,2\}, \\ 1 & \text{if } d_H(x) = 0. \end{cases}$

Ex. Isolated vertex gives 1, edge 3, triangle 6.

Lem. To prove $\mathring{s}(G) \leq \frac{7}{3}|V(G)|$ for outerplanar G, show $\mathring{s}(H) \leq \Phi(H)$ for induced subgr of max'l outerplanar H.

Sketch for Outerplanar Graphs

For an induced subgraph H of a maximal outerplanar graph, define potential $\Phi(H) = \sum_{x \in V(H) \cup E(H)} \phi(x, H)$,

where $\phi(x, H) = \begin{cases} 1/3 & \text{if } x \in E(H), \\ 5/3 & \text{if } x \text{ is a vertex in a triangle, else} \\ 5/3 & \text{if } d_H(x) \geq 3, \\ 4/3 & \text{if } d_H(x) \in \{1, 2\}, \\ 1 & \text{if } d_H(x) = 0. \end{cases}$

Ex. Isolated vertex gives 1, edge 3, triangle 6.

Lem. To prove $\mathring{s}(G) \leq \frac{7}{3}|V(G)|$ for outerplanar G, show $\mathring{s}(H) \leq \Phi(H)$ for induced subgr of max'l outerplanar H.

Pf. Maximal outerplanar has $2n - 3$ edges and every vertex in a triangle, so $\Phi(H) = \frac{5}{3}n + \frac{1}{3}(2n - 3) < \frac{7}{3}n$. ■

Sketch for Outerplanar Graphs

For an induced subgraph H of a maximal outerplanar graph, define potential $\Phi(H) = \sum_{x \in V(H) \cup E(H)} \phi(x, H)$,

where $\phi(x,H) = \begin{cases} 1/3 & \text{if } x \in E(H), \\ 5/3 & \text{if } x \text{ is a vertex in a triangle, else} \\ 5/3 & \text{if } d_H(x) \geq 3, \\ 4/3 & \text{if } d_H(x) \in \{1, 2\}, \\ 1 & \text{if } d_H(x) = 0. \end{cases}$

Ex. Isolated vertex gives 1, edge 3, triangle 6.

Lem. To prove $\mathring{s}(G) \leq \frac{7}{3}|V(G)|$ for outerplanar G, show $\mathring{s}(H) \leq \Phi(H)$ for induced subgr of max'l outerplanar H.

Pf. Maximal outerplanar has $2n-3$ edges and every vertex in a triangle, so $\Phi(H) = \frac{5}{3}n + \frac{1}{3}(2n-3) < \frac{7}{3}n$. ■

Idea: For M in an induced subgraph G of a maximal outerplanar graph, Painter seeks an independent set X such that $|M| \leq \Phi(G) - \Phi(G - X)$; inductively this suffices.

How to Find X?

We want independent $X \subseteq M$ with $|M| \leq \Phi(G) - \Phi(G - X)$.

How to Find X?

We want independent $X \subseteq M$ with $|M| \leq \Phi(G) - \Phi(G - X)$.

Let utility $u(X) = \Phi(G) - \Phi(G - X) - |M|$. Want $u(X) \geq 0$.

How to Find X?

We want independent $X \subseteq M$ with $|M| \leq \Phi(G) - \Phi(G - X)$.

Let utility $u(X) = \Phi(G) - \Phi(G - X) - |M|$. Want $u(X) \geq 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v, G) + \frac{1}{3} \cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v, G) - \phi(v, G - X) - 1 & \text{if } v \in M - X, \\ \phi(v, G) - \phi(v, G - X) & \text{if } v \in V(G) - M. \end{cases}$$

How to Find X?

We want independent $X \subseteq M$ with $|M| \leq \Phi(G) - \Phi(G - X)$.

Let utility $u(X) = \Phi(G) - \Phi(G - X) - |M|$. Want $u(X) \geq 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v, G) + \frac{1}{3} \cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v, G) - \phi(v, G - X) - 1 & \text{if } v \in M - X, \\ \phi(v, G) - \phi(v, G - X) & \text{if } v \in V(G) - M. \end{cases}$$

(Recall $\phi(x, G) = 1/3$ if $x \in E(G)$, and X is independent.)

How to Find X?

We want independent $X \subseteq M$ with $|M| \leq \Phi(G) - \Phi(G - X)$.

Let utility $u(X) = \Phi(G) - \Phi(G - X) - |M|$. Want $u(X) \geq 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v, G) + \frac{1}{3} \cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v, G) - \phi(v, G - X) - 1 & \text{if } v \in M - X, \\ \phi(v, G) - \phi(v, G - X) & \text{if } v \in V(G) - M. \end{cases}$$

(Recall $\phi(x, G) = 1/3$ if $x \in E(G)$, and X is independent.)

Idea: Cover M using three independent sets A, B, C (some vertices twice) such that $u(A) + u(B) + u(C) \geq 0$.

How to Find X?

We want independent $X \subseteq M$ with $|M| \leq \Phi(G) - \Phi(G - X)$.

Let utility $u(X) = \Phi(G) - \Phi(G - X) - |M|$. Want $u(X) \geq 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v, G) + \frac{1}{3} \cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v, G) - \phi(v, G - X) - 1 & \text{if } v \in M - X, \\ \phi(v, G) - \phi(v, G - X) & \text{if } v \in V(G) - M. \end{cases}$$

(Recall $\phi(x, G) = 1/3$ if $x \in E(G)$, and X is independent.)

Idea: Cover M using three independent sets A, B, C (some vertices twice) such that $u(A) + u(B) + u(C) \geq 0$.

Note: $u_X(v) \geq 0$ when $v \notin M$ and $u_X(v) \geq -1$ when $v \in M$.

How to Find X?

We want independent $X \subseteq M$ with $|M| \le \Phi(G) - \Phi(G-X)$.

Let utility $u(X) = \Phi(G) - \Phi(G-X) - |M|$. Want $u(X) \ge 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v,G) + \frac{1}{3} \cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v,G) - \phi(v,G-X) - 1 & \text{if } v \in M - X, \\ \phi(v,G) - \phi(v,G-X) & \text{if } v \in V(G) - M. \end{cases}$$

(Recall $\phi(x,G) = 1/3$ if $x \in E(G)$, and X is independent.)

Idea: Cover M using three independent sets A, B, C (some vertices twice) such that $u(A) + u(B) + u(C) \ge 0$.

Note: $u_X(v) \ge 0$ when $v \notin M$ and $u_X(v) \ge -1$ when $v \in M$.

Start with coloring A, B, C; some vertices get extra color.

How to Find X?

We want independent $X \subseteq M$ with $|M| \le \Phi(G) - \Phi(G-X)$.

Let utility $u(X) = \Phi(G) - \Phi(G-X) - |M|$. Want $u(X) \ge 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v,G) + \frac{1}{3}\cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v,G) - \phi(v,G-X) - 1 & \text{if } v \in M - X, \\ \phi(v,G) - \phi(v,G-X) & \text{if } v \in V(G) - M. \end{cases}$$

(Recall $\phi(x,G) = 1/3$ if $x \in E(G)$, and X is independent.)

Idea: Cover M using three independent sets A, B, C (some vertices twice) such that $u(A) + u(B) + u(C) \ge 0$.

Note: $u_X(v) \ge 0$ when $v \notin M$ and $u_X(v) \ge -1$ when $v \in M$.

Start with coloring A, B, C; some vertices get extra color.

- $u_A(v) + u_B(v) + u_C(v) \ge 0$ for $v \in M$ except in some paths.

How to Find X?

We want independent $X \subseteq M$ with $|M| \le \Phi(G) - \Phi(G-X)$.

Let utility $u(X) = \Phi(G) - \Phi(G-X) - |M|$. Want $u(X) \ge 0$.

Apportion $u(X)$ to vertices (utility of v relative to X):

$$u_X(v) = \begin{cases} \phi(v,G) + \frac{1}{3}\cdot d_G(v) - 1 & \text{if } v \in X, \\ \phi(v,G) - \phi(v,G-X) - 1 & \text{if } v \in M - X, \\ \phi(v,G) - \phi(v,G-X) & \text{if } v \in V(G) - M. \end{cases}$$

(Recall $\phi(x,G) = 1/3$ if $x \in E(G)$, and X is independent.)

Idea: Cover M using three independent sets A, B, C (some vertices twice) such that $u(A) + u(B) + u(C) \ge 0$.

Note: $u_X(v) \ge 0$ when $v \notin M$ and $u_X(v) \ge -1$ when $v \in M$.

Start with coloring A, B, C; some vertices get extra color.

- $u_A(v) + u_B(v) + u_C(v) \ge 0$ for $v \in M$ except in some paths.
- $\sum_{v \in V(P)} [u_A(v) + u_B(v) + u_C(v)] \ge 0$ for those paths. ■

Open Problems

Open Problems

Ques. What are the best bounds for n-vertex outerplanar, planar, or d-degenerate graphs?

Open Problems

Ques. What are the best bounds for n-vertex outerplanar, planar, or d-degenerate graphs?

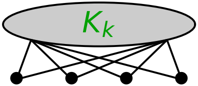
Ques. Do the results on trees generalize to k-trees?

Open Problems

Ques. What are the best bounds for n-vertex outerplanar, planar, or d-degenerate graphs?

Ques. Do the results on trees generalize to k-trees?

We know $\mathring{s}(K_k \oplus \overline{K}_r) = r + \binom{k+1}{2} + k u_r$. Minimum?

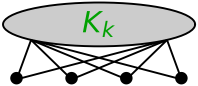


Open Problems

Ques. What are the best bounds for n-vertex outerplanar, planar, or d-degenerate graphs?

Ques. Do the results on trees generalize to k-trees?

We know $\mathring{s}(K_k \oplus \overline{K}_r) = r + \binom{k+1}{2} + ku_r$. Minimum?



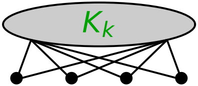
We know $\mathring{s}(P_n^k) \geq \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k+2}{2} + \binom{r+1}{2}$, Maximum?

Open Problems

Ques. What are the best bounds for n-vertex outerplanar, planar, or d-degenerate graphs?

Ques. Do the results on trees generalize to k-trees?

We know $\mathring{s}(K_k \diamondplus \overline{K}_r) = r + \binom{k+1}{2} + ku_r$. Minimum?



We know $\mathring{s}(P_n^k) \geq \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k+2}{2} + \binom{r+1}{2}$, Maximum?

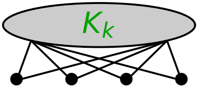
Ques. Complexity? Algorithm for outerplanar?

Open Problems

Ques. What are the best bounds for n-vertex outerplanar, planar, or d-degenerate graphs?

Ques. Do the results on trees generalize to k-trees?

We know $\mathring{s}(K_k \oplus \overline{K}_r) = r + \binom{k+1}{2} + ku_r$. Minimum?



We know $\mathring{s}(P_n^k) \geq \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k+2}{2} + \binom{r+1}{2}$, Maximum?

Ques. Complexity? Algorithm for outerplanar?

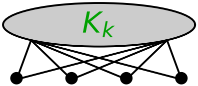
Ques. Best bound for bipartite planar?
Puleo proved $\mathring{s}(C_4 \square P_k) \geq 7k - 1 \sim 1.75n$.

Open Problems

Ques. What are the best bounds for n-vertex outerplanar, planar, or d-degenerate graphs?

Ques. Do the results on trees generalize to k-trees?

We know $\mathring{s}(K_k \oplus \overline{K}_r) = r + \binom{k+1}{2} + k u_r$. Minimum?



We know $\mathring{s}(P_n^k) \geq \left\lfloor \frac{n}{k+1} \right\rfloor \binom{k+2}{2} + \binom{r+1}{2}$, Maximum?

Ques. Complexity? Algorithm for outerplanar?

Ques. Best bound for bipartite planar?
Puleo proved $\mathring{s}(C_4 \,\square\, P_k) \geq 7k - 1 \sim 1.75n$.

Ques. Generalize the result on sparse 4-colorable graphs to sparse k-colorable graphs.