On the Bishop frame of a partially null curve in Minkowski spacetime

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Introduction

Bishop frame or relatively parallel adapted frame $\{T, N_1, N_2\}$ of a regular curve α in Euclidean space $\mathbb{E}^3$ is introduced by Richard Bishop in 1975. It contains tangential vector field T and two normal vector fields N_1 and N_2 obtained by rotating the principal normal vector field N and the binormal vector field B of α in the normal plane $T^{\perp}$ for the corresponding angle θ.

After such rotation, the derivatives N_1' and N_2' are collinear with the tangent vector field T of the curve, so they make no rotations in the planes $\{T, N_2\}$ and $\{T, N_1\}$ respectively. This is why the Bishop frame is also called rotation-minimizing frame.

The vector fields N_1 and N_2, whose derivatives N_1' and N_2' are always collinear with T, are called relatively parallel vector fields.

The Bishop frame of a regular curve in Euclidean 3-space is not unique.

Introduction

The Bishop frame can be used in many physical and mathematical applications related with:

- rigid body mechanics;
- computer graphics;
- deformation of tubes;
- sweep surface modeling;
- differential geometry in studying different types of curves.

The relation between the Bishop frame $\{T, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of a regular curve in E^3 is given by

$$\begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\theta(s) = \int \tau(s)\, ds$ is an angle of rotation and $\tau(s)$ is the torsion.

Introduction

The Bishop frame's equations read

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

The Bishop curvatures of a regular curve α in E^3 have the form

$$\kappa_1(s) = \kappa(s)\cos\theta(s), \quad \kappa_2(s) = \kappa(s)\sin\theta(s),$$

where $\kappa(s)$ is the first Frenet curvature.

Introduction

Minkowski spacetime $\mathbb{E}_1^4$ is the real vector space $\mathbb{E}^4$ equipped with the standard indefinite flat metric $\langle \cdot, \cdot \rangle$ defined by

$$\langle V, W \rangle = -v_1 w_1 + v_2 w_2 + v_3 w_3 + v_4 w_4,$$

for any two vectors $V = (v_1, v_2, v_3, v_4)$ and $W = (w_1, w_2, w_3, w_4)$ in $\mathbb{E}_1^4$.

A vector V in $\mathbb{E}_1^4$ can be spacelike if $\langle V, V \rangle > 0$ or $V = 0$, timelike if $\langle V, V \rangle < 0$, and null (lightlike), if $\langle V, V \rangle = 0$ and $V \neq 0$.

The curve α can locally be spacelike, timelike or null (lightlike), if its velocity vectors are spacelike, timelike or null, respectively.

A spacelike curve in E_1^4 is called pseudo null curve, if its principal normal and the second binormal vector fields are null.

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A null curve in E_1^4 is called null Cartan curve, if it is parameterized by pseudo-arc function defined by

$$s(t) = \int_0^t \sqrt{||\alpha''(u)||}\, du.$$

A spacelike curve in E_1^4 is called partially null curve, if its the first and the second binormal vector fields are null.
The Frenet frame $\{T, N, B_1, B_2\}$ of a partially null curve satisfies the conditions

$$\langle T, T\rangle = \langle N, N\rangle = \langle B_1, B_2\rangle = 1,\ \langle B_1, B_1\rangle = \langle B_2, B_2\rangle = 0,$$
$$\langle T, N\rangle = \langle T, B_1\rangle = \langle T, B_2\rangle = \langle N, B_1\rangle = \langle N, B_2\rangle = 0.$$

Partially null curve lies in a lightlike hyperplane of E_1^4 and has the curvature functions $k_1 \neq 0$, $k_2 \neq 0$, $k_3 = 0$.

Introduction

The Frenet equations of partially null curve read

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

The vector fields T', B_1' and B_2' are always collinear with N, so they make minimal rotations along partially null curve.

Therefore, the Frenet frame of partially null curve has the rotation minimizing property.

Now we can ask the next question is: Is there some another frame of a partially null curve, different than Frenet frame, having a rotation minimizing property?

Some known results

In Euclidean 4-space E^4, the relation between Bishop frame $\{T, N_1, N_2, N_3\}$ and Frenet frame $\{T, N, B_1, B_2\}$ of a regular curve is obtained in 2014 and has the form

$$\begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f_{22} & f_{23} & f_{24} \\ 0 & f_{32} & f_{33} & f_{34} \\ 0 & f_{42} & f_{43} & f_{44} \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where

$$f_{22} = \cos\theta\cos\psi, \quad f_{23} = -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi,$$

$$f_{24} = \sin\phi\sin\psi + \cos\phi\sin\theta\sin\psi,$$

$$f_{32} = \cos\theta\sin\psi, \quad f_{33} = \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi,$$

$$f_{34} = -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi,$$

Some known results

$$f_{42} = -\sin\theta, \quad f_{43} = \sin\phi\cos\theta, \quad f_{44} = \cos\phi\cos\theta.$$

and where the angles θ, ψ, ϕ are Euler's angles.
The Bishop frame's equations read

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \\ N_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1(s) & \kappa_2(s) & \kappa_3(s) \\ -\kappa_1(s) & 0 & 0 & 0 \\ -\kappa_2(s) & 0 & 0 & 0 \\ -\kappa_3(s) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \\ N_3(s) \end{bmatrix}.$$

The Bishop curvatures of a regular curve in E^4 are given by

$$\kappa_1 = \kappa\cos\theta\cos\psi, \quad \kappa_2 = \kappa(-\cos\theta\sin\psi + \sin\phi\sin\theta\cos\psi),$$

$$\kappa_3 = \kappa(\sin\theta\sin\psi + \cos\phi\sin\theta\cos\psi).$$

Some known results

If α is spacelike curve with timelike principal normal in E_1^4, the Bishop frame's equations read (2015, M. Erdogdu)

$$\begin{bmatrix} T'(s) \\ N_1'(s) \\ N_2'(s) \\ N_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1(s) & \kappa_2(s) & \kappa_3(s) \\ \kappa_1(s) & 0 & 0 & 0 \\ -\kappa_2(s) & 0 & 0 & 0 \\ -\kappa_3(s) & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \\ N_3(s) \end{bmatrix}.$$

In this case, the Bishop curvatures have the form

$$\kappa_1 = \kappa \cosh\theta \cosh\psi,$$

$$\kappa_2 = \kappa(\cosh\theta \sinh\psi + \sinh\phi \sinh\theta \cosh\psi),$$

$$\kappa_3 = \kappa(\sinh\theta \sinh\psi + \cosh\phi \sinh\theta \cosh\psi),$$

and where θ, ψ and ϕ are hyperbolic Euler's angles.

Some known results

If α is null Cartan curve in E_1^4, the Bishop frame $\{T_1, N_1, N_2, N_3\}$ and the Cartan frame $\{T, N, B_1, B_2\}$ of α are related by (2018, K. Ilarslan, E. Nešović)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\sigma_1\sigma_2 - \sigma_3\sqrt{\sigma_1'^2 + \sigma_3'^2} & \sigma_1 & 0 & \sigma_3 \\ \dfrac{\sigma_2^2 + \sigma_1'^2 + \sigma_3'^2}{2} & -\sigma_2 & 1 & -\sqrt{\sigma_1'^2 + \sigma_3'^2} \\ \sigma_2\sigma_3 - \sigma_1\sqrt{\sigma_1'^2 + \sigma_3'^2} & -\sigma_3 & 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

The Bishop curvatures of null Cartan curve have the form

$$\sigma_1(s) = \sin\theta(s), \quad \sigma_3(s) = \cos\theta(s),$$

$\sigma_2(s)$ satisfies the differential equation

$$\sigma_2(s) = \frac{\kappa_3(s) - \theta''(s)}{\theta'(s)}, \quad \theta'(s) \neq 0,$$

Some known results

where the function θ satisfies the third order non-linear differential equation

$$2\theta'(\theta''' - \kappa_3') + 2\theta''(\kappa_3 - \theta'') + \theta'^4 - (\kappa_3 - \theta'')^2 - 2\kappa_2\theta'^2 = 0,$$

and κ_2 and κ_3 are the second and the third Cartan curvature of the curve.

In this case, the Bishop frame's equations read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} \sigma_2 & \sigma_1 & 0 & -\sigma_3 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & -\sigma_3 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

Some known results

If α is a pseudo null curve in E_1^4, the relation between Bishop frame $\{N_0, N_1, N_2, N_3\}$ and Frenet frame reads (2020, J. Djordjević, E. Nešović)

$$\begin{aligned}
N_0 &= \frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}T + \Big(\frac{(\frac{\sigma_2}{\sigma_1})'\sigma_1^2\sigma_2}{(\sigma_1^2+\sigma_2^2)^2} - \frac{\sigma_1\sigma_3}{\sigma_1^2+\sigma_2^2}\Big)N + \frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}B_1,\\
N_1 &= N,\\
N_2 &= -\frac{\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}T - \Big(\frac{(\frac{\sigma_2}{\sigma_1})'\sigma_1^3}{(\sigma_1^2+\sigma_2^2)^2} + \frac{\sigma_2\sigma_3}{\sigma_1^2+\sigma_2^2}\Big)N + \frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}B_1,\\
N_3 &= -\frac{(\frac{\sigma_2}{\sigma_1})'\sigma_1^2}{(\sigma_1^2+\sigma_2^2)^{\frac{3}{2}}}T - \frac{1}{2}\Big(\frac{\sigma_3^2}{\sigma_1^2+\sigma_2^2} + \frac{(\frac{\sigma_2}{\sigma_1})'^{\,2}\sigma_1^4}{(\sigma_1^2+\sigma_2^2)^3}\Big)N + \frac{\sigma_3}{\sqrt{\sigma_1^2+\sigma_2^2}}B_1 + B_2.
\end{aligned}$$

The Bishop curvatures σ_1, σ_2 and σ_3 of α have the form

$$\begin{aligned}
\sigma_1(s) &= \kappa_2(s)\cos\theta(s), \quad \sigma_2(s) = \kappa_2(s)\sin\theta(s),\\
\sigma_3(s) &= -\frac{\kappa_2(s)}{\theta'(s)}\Big(1 + \Big(\frac{\theta'(s)}{\kappa_2(s)}\Big)'\Big),
\end{aligned}$$

Some known results

where the function θ satisfies the third order non-linear differential equation

$$\left(\frac{1}{\theta'}+\frac{1}{\theta'}\Big(\frac{\theta'}{\kappa_2}\Big)'\right)'-\frac{\kappa_2}{2}\left(\frac{1}{\theta'}+\frac{1}{\theta'}\Big(\frac{\theta'}{\kappa_2}\Big)'\right)^2+\frac{\theta'^{\,2}}{2\kappa_2}+\kappa_3=0,$$

and κ_2 and κ_3 are the second and the third Frenet curvature of the curve.

In this case, the Bishop frame's equations read

$$\begin{bmatrix} N_0' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -\sigma_1 \\ \sigma_1 & \sigma_3 & \sigma_2 & 0 \\ 0 & 0 & 0 & -\sigma_2 \\ 0 & 0 & 0 & -\sigma_3 \end{bmatrix} \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

Bishop frame of partially null curve in E_1^4

Definition 1. The Bishop frame $\{T_1, N_1, N_2, N_3\}$ of a partially null curve in $\mathbb{E}_1^4$ is positively oriented pseudo-orthonormal frame containing the second binormal vector field $N_3 = B_2$, relatively parallel spacelike vector fields T_1 and N_1 and relatively parallel null vector field N_2 satisfying the conditions

$$\langle T_1, T_1 \rangle = \langle N_1, N_1 \rangle = \langle N_2, N_3 \rangle = 1, \ \langle N_2, N_2 \rangle = \langle N_3, N_3 \rangle = 0,$$
$$\langle T_1, N_1 \rangle = \langle T_1, N_2 \rangle = \langle T_1, N_3 \rangle = \langle N_1, N_2 \rangle = \langle N_1, N_3 \rangle = 0.$$

Definition 2. The vector fields T_1, N_1 and N_2 are relatively parallel, if their derivatives' component spanned by $\{T_1, N_1, N_3\}$ is equal to zero.

This means that the derivatives T_1', N_1' and N_2' are always collinear with vector field N_2.

Bishop frame of partially null curve in E_1^4

Theorem 1. Let α be a partially null curve in E_1^4. Then:
(A) the relation between Bishop frame $\{T_1, N_1, N_2, N_3\}$ and Frenet frame $\{T, N, B_1, B_2\}$ reads

$$
\begin{aligned}
&T_1 = -\sin\theta T - \cos\theta N + [\tfrac{\theta' - k_1}{k_2}\sin\theta - \cos\theta \frac{(\frac{k_1-\theta'}{k_2})'}{\theta'}]B_2,\\
&N_1 = \cos\theta T - \sin\theta N - [\tfrac{\theta' - k_1}{k_2}\cos\theta + \sin\theta \frac{(\frac{k_1-\theta'}{k_2})'}{\theta'}]B_2,\\
&N_2 = \tfrac{\theta' - k_1}{k_2}T - \frac{(\frac{k_1-\theta'}{k_2})'}{\theta'}N + B_1 - \tfrac{1}{2}[(\tfrac{\theta' - k_1}{k_2})^2 + \frac{(\frac{k_1-\theta'}{k_2})'}{\theta'}]B_2,\\
&N_3 = B_2.
\end{aligned}
$$

(B) The Bishop curvatures σ_1, σ_2 and σ_3 of α have the form

$$
\begin{aligned}
\sigma_1(s) &= \kappa_2(s)\cos\theta(s), \quad \sigma_2(s) = \kappa_2(s)\sin\theta(s),\\
\sigma_3(s) &= \frac{k_2(\frac{k_1-\theta'}{k_2})'}{\theta'}
\end{aligned}
$$

where the function θ satisfies the third order non-linear differential equation

$$(\frac{\theta' - k_1}{k_2})\theta' - (\frac{(\frac{k_1-\theta'}{k_2})'}{\theta'})' = \frac{k_2}{2}(\frac{\theta' - k_1}{k_2})^2 + k_2(\frac{(\frac{k_1-\theta'}{k_2})'}{\theta'})^2. \quad (1)$$

(C) The Bishop frame's equations read

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sigma_1 & 0 \\ 0 & 0 & -\sigma_2 & 0 \\ 0 & 0 & -\sigma_3 & 0 \\ \sigma_1 & \sigma_2 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}.$$

Remark. The Bishop frame of partially null curve is not unique. Every particular solution of the above differential equation (1) provides the corresponding Bishop frame and Bishop curvatures.

Bishop frame of partially null curve in E_1^4

Example. Let us consider partially null curve α in $\mathbb{E}_1^4$ with parametric equation

$\alpha(s) = (\sin s + \cos s - 2s, \sin s + \cos s - 2s, \cos s, -\cos(s), \sin s).$

The Frenet curvatures of α read

$$\kappa_1(s) = 1, \quad \kappa_2(s) = -\frac{8}{3}, \quad \kappa_3(s) = 0.$$

Substituting the Frenet curvatures in differential equation (1), we obtain new differential equation

$$-\frac{3}{8}(\theta' - 1)\theta' - \frac{3}{8}(\frac{\theta''}{\theta'})' = -\frac{9}{48}(\theta' - 1)^2 + (\frac{\theta''}{\theta'})^2. \qquad (2)$$

with particular solution $\theta(s) = -s$.

Bishop frame of partially null curve in E_1^4

The Bishop curvatures, that correspond to particular solution $\theta(s) = -s$, are given by

$$\sigma_1(s) = -\frac{8}{3}\cos(s), \quad \sigma_2(s) = \frac{8}{3}\sin(s), \quad \sigma_3(s) = 0.$$

According to Theorem 1, the Bishop frame of α reads

$$\begin{aligned}
T_1(s) &= \sin(s)T(s) - \cos(s)N(s) - \frac{3}{4}\sin(s)B_2(s),\\
N_1(s) &= \cos(s)T(s) + \sin(s)N(s) - \frac{3}{4}\cos(s)B_2(s),\\
N_2(s) &= \frac{3}{4}T(s) + B_1(s) - \frac{9}{32}B_2(s),\\
N_3(s) &= B_2(s).
\end{aligned}$$

Bishop frame of partially null curve in E_1^4

Another particular solution of differential equation (1) is $\theta(s) = s$. The Bishop curvatures, that correspond to particular solution $\theta(s) = s$, have the forms

$$\sigma_1(s) = -\frac{8}{3}\cos(s), \quad \sigma_2(s) = -\frac{8}{3}\sin(s), \quad \sigma_3(s) = 0.$$

According to Theorem 1, the Bishop frame of α in this case reads

$$\begin{aligned} T_1(s) &= -\sin(s)T(s) - \cos(s)N(s), \\ N_1(s) &= \cos(s)T(s) - \sin(s)N(s), \\ N_2(s) &= B_1(s), \\ N_3(s) &= B_2(s). \end{aligned}$$

Darboux bivectors of the Frenet and Bishop frame of partially null curve in E_1^4

Spacetime algebra is geometric algebra $G_4(\mathbb{E}_1^4)$ of dimension 16. Its elements are called multivectors. In spacetime algebra, the geometric product of a vector a with itself is defined by

$$aa = a^2 = \epsilon_a ||a||^2,$$

where ϵ_a is the signature of a and $||a|| = \sqrt{|\langle a, a \rangle|}$ is a magnitude of a.
The geometric product of vectors a and b has the next decomposition

$$ab = a \cdot b + a \wedge b.$$

in terms of inner product $a \cdot b = \frac{1}{2}(ab + ba)$ and outer (wedge) product $a \wedge b = \frac{1}{2}(ab - ba)$.

Darboux bivectors of the Frenet and Bishop frame of partially null curve in E_1^4

The inner product of 1-vector a and 2-vector $b \wedge c$ is given by equation

$$a \cdot (b \wedge c) = \langle a, b \rangle c - \langle a, c \rangle b = -(b \wedge c) \cdot a.$$

The inner product of two 2-vectors $b \wedge a$ and $u \wedge v$ is defined by

$$(b \wedge a) \cdot (u \wedge v) = -\langle b, u \rangle \langle a, v \rangle + \langle b, v \rangle \langle a, u \rangle.$$

Darboux bivectors of the Frenet and Bishop frame of partially null curve in E_1^4

The Darboux bivector D of the Frenet frame of partially null curve in E_1^4 can be written as the sum of its projections on six 2-planes parallel to basis bivectors

$$D = aE_{TN} + bE_{TB_1} + cE_{TB_2} + dE_{NB_1} + fE_{NB_2} + hE_{B_1B_2},$$

where the coefficients a, b, c, d, f, h are areas of the projections.

Darboux bivector D satisfies the Darboux equations of the form

$$T' = D \cdot T, \quad N' = D \cdot N, \quad B_1' = D \cdot B_1, \quad B_2' = D \cdot B_2.$$

By using Darboux equations, it follows that the Darboux bivector D has the equation

$$D = -\kappa_1 E_{TN} - \kappa_2 E_{NB_1}.$$

Darboux bivectors of the Frenet and Bishop frame of partially null curve in E_1^4

The Frenet curvatures

$$\kappa_1 = D \cdot E_{TN}, \quad \kappa_2 = D \cdot E_{NB_2}$$

can be interpreted as areas of the projections of Darboux bivector D onto the spacelike 2-plane $span\{T, N\}$ and lightlike 2-plane $span\{N, B_1\}$ respectively.

Similarly, Darboux bivector $\bar{D}$ of the Bishop frame can be written as the sum of its projections on six 2-planes parallel to basis bivectors

$$\bar{D} = aE_{T_1N_1} + bE_{T_1N_2} + cE_{T_1N_3} + dE_{N_1N_2} + fE_{N_1N_3} + hE_{N_2N_3},$$

where the coefficients a, b, c, d, f, h are areas of the projections.

Darboux bivectors of the Frenet and Bishop frame of partially null curve in E_1^4

By using Darboux equations, it follows that Darboux bivector of the Bishop frame has the equation

$$\bar{D} = \sigma_1 E_{T_1 N_2} + \sigma_2 E_{N_1 N_2} - \sigma_3 E_{N_2 N_3}.$$

The Bishop curvatures

$$\sigma_1 = -\bar{D} \cdot E_{T_1 N_3}, \quad \sigma_2 = -\bar{D} \cdot E_{N_1 N_3}, \quad \sigma_3 = -\bar{D} \cdot E_{N_2 N_3},$$

can be interpreted as areas of the projections of Darboux bivector $\bar{D}$ onto the lightlike 2-plane $span\{T_1, N_2\}$, lightlike 2-plane $span\{N_1, N_2\}$ and timelike 2-plane $span\{N_2, N_3\}$, respectively.

THANK YOU!