Real Cayley-Dickson algebras: doubly alternative elements and zero divisor graphs

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Orthogonality and zero divisor graphs

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$, $Z(\mathcal{A})$ be the set of its zero divisors, $Z_{LR}(\mathcal{A})$ be the set of its two-sided zero divisors. We introduce the following relation graphs of $\mathcal{A}$.

**Definition 1**

*The orthogonality graph* $\Gamma_O(\mathcal{A})$:

- the vertex set is $P(Z_{LR}(\mathcal{A})) = \{[a] = \mathbb{F}^* a \mid a \in Z_{LR}(\mathcal{A})\}$,
- $[a]$ and $[b]$ are connected $\iff [a] \neq [b], \ ab = ba = 0$.

*The directed zero divisor graph* $\Gamma_Z(\mathcal{A})$:

- the vertex set is $P(Z(\mathcal{A})) = \{[a] = \mathbb{F}^* a \mid a \in Z(\mathcal{A})\}$,
- there is an edge from $[a]$ to $[b] \iff [a] \neq [b], \ ab = 0$.

The edges of $\Gamma_O(\mathcal{A})$ and $\Gamma_Z(\mathcal{A})$ are well-defined. When speaking of the vertices of these graphs, we will not distinguish between a nonzero element $a$ and a line $[a] = \mathbb{F}a$ passing through it.
Definition 2

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ with an involution $a \mapsto \bar{a}$. The algebra $\mathcal{A}\{\gamma\}$ produced by the Cayley-Dickson process, when applied to $\mathcal{A}$ with the parameter $\gamma \in \mathbb{F}$, $\gamma \neq 0$, is defined as the set of ordered pairs of elements of $\mathcal{A}$ with operations

$$\alpha(a, b) = (\alpha a, \alpha b);$$

$$(a, b) + (c, d) = (a + c, b + d);$$

$$(a, b)(c, d) = (ac + \gamma \bar{d}b, da + b\bar{c})$$

and the involution

$$(a, b) = (\bar{a}, -b), \quad a, b, c, d \in \mathcal{A}, \quad \alpha \in \mathbb{F}.$$ 

If $\mathcal{A}$ is unital and the involution on $\mathcal{A}$ is regular, that is, $a + \bar{a} \in \mathbb{F}1_{\mathcal{A}}$ and $a\bar{a} = \bar{a}a \in \mathbb{F}1_{\mathcal{A}}$ for all $a \in \mathcal{A}$, then the involution on $\mathcal{A}\{\gamma\}$ is also regular.
We now assume that $F = \mathbb{R}$.

**Definition 3**

Let $n \in \mathbb{N}_0$. The algebra

$$\mathcal{A}_n = \mathcal{A}_n\{\gamma_0, \ldots, \gamma_{n-1}\} = (\ldots (\mathbb{R}\{\gamma_0\}) \ldots)\{\gamma_{n-1}\}$$

is a real Cayley-Dickson algebra determined by the parameters $\gamma_0, \ldots, \gamma_{n-1} \in \mathbb{R} \setminus \{0\}$.

In other words, $\mathcal{A}_n$ is defined recursively by using the identities $\mathcal{A}_0 = \mathbb{R}$ and $\mathcal{A}_n\{\gamma_0, \ldots, \gamma_{n-1}\} = (\mathcal{A}_{n-1}\{\gamma_0, \ldots, \gamma_{n-2}\})\{\gamma_{n-1}\}$. Clearly, $\dim \mathcal{A}_n = 2^n$.

**Proposition 4**

$Z(\mathcal{A}_n) = Z_{LR}(\mathcal{A}_n)$. 

Real Cayley-Dickson algebras
\( A_n\{\gamma_0, \ldots, \gamma_{n-1}\} \) is isomorphic to \( A_n\{\text{sgn}(\gamma_0), \ldots, \text{sgn}(\gamma_{n-1})\} \), so it is sufficient to consider \( \gamma_k \in \{\pm 1\} \) only, \( k = 0, \ldots, n-1 \).

- If \( \gamma_0 = \cdots = \gamma_{n-1} = -1 \) then \( A_n = M_n = M_{n-1}\{-1\} \) is an algebra of the main sequence.
  - The complex numbers: \( \mathbb{C} \cong M_1; \)
  - The quaternions: \( \mathbb{H} \cong M_2; \)
  - The octonions: \( \mathbb{O} \cong M_3; \)
  - The sedenions: \( \mathbb{S} \cong M_4; \)
- If \( \gamma_0 = \cdots = \gamma_{n-2} = -1 \) and \( \gamma_{n-1} = 1 \) then \( A_n = H_n = M_{n-1}\{1\} \) is a Cayley-Dickson split-algebra.
  - The split-complex numbers: \( \hat{\mathbb{C}} \cong H_1; \)
  - The split-quaternions: \( \hat{\mathbb{H}} \cong H_2; \)
  - The split-octonions: \( \hat{\mathbb{O}} \cong H_3; \)
  - The split-sedenions: \( \hat{\mathbb{S}} \cong H_4. \)
The real part and the norm

For any $a \in \mathcal{A}_n$ we define

- **real part:** $\Re(a) = \frac{a + \bar{a}}{2}$;
- **imaginary part:** $\Im(a) = \frac{a - \bar{a}}{2}$;
- **norm:** $n(a) = a\bar{a} = \bar{a}a$.

**Proposition 5**

We can compute real part and norm of an element $(a, b) \in \mathcal{A}_{n+1}$ inductively by using the following equalities:

$$\Re((a, b)) = \Re(a),$$
$$n((a, b)) = n(a) - \gamma_n n(b).$$

**Definition 6**

An element $(a, b) \in \mathcal{A}_{n+1}$ is said to be

- **pure** if $\Re(a) = 0$,
- **doubly pure** if $\Re(a) = \Re(b) = 0$. 

Alternative elements

- The associator of \( a, b, c \in A \) is \([a, b, c] = (ab)c - a(bc)\).
- \( A \) is called *flexible* if \([a, b, a] = 0\) for all \( a, b \in A \).
- If \( A \) is flexible then \([a, b, c] = -[c, b, a]\) for all \( a, b, c \in A \).
- \( A \) is called *alternative* if \([a, a, b] = [b, a, a] = 0\) for all \( a, b \in A \).
- It is well known that \( A_n \) is alternative if and only if \( n \leq 3 \), however, \( A_n \) is always flexible.

**Definition 7 (Moreno, 2006)**

Let \( a, b \in A_n \).

- We say that \( a \) *alternates* with \( b \) if \([a, a, b] = 0\).
- If \( a \) alternates with every \( b \in A_n \) then \( a \) is *alternative*.
- We say that \( a \) *alternates strongly* with \( b \) if \([a, a, b] = 0\) and \([b, b, a] = 0\).
- If \( a \) alternates strongly with every \( b \in A_n \) then \( a \) is *strongly alternative*.
Zero divisors with restrictions on norm and alternativity

We now assume that \( a, b \in A_n \) alternate strongly with \( c, d \in A_n \),
\((a, b)(c, d) = 0 \) in \( A_{n+1} \).

**Lemma 8**

Let \( n(c) - \chi \gamma_n n(d) = \chi n(c) - \gamma_n n(d) = 0 \) for some \( \chi \in \mathbb{R} \). Then
\( (c, d)(\overline{ac}, -\chi da) = 0 \).

**Proof.**

Indeed, we have
\[
(c, d)(\overline{ac}, -\chi da) = (c(\overline{ac}) + \gamma_n(\overline{\chi da}d, (-\chi da)c + d(ac)) = \\
= (c(\overline{\chi a}) - \chi \gamma_n(\overline{d}d, \chi(b\overline{c})c - \gamma_n d(\overline{d}b)) = \\
= (((c\overline{c})\overline{a} - \chi \gamma_n \overline{a}(\overline{d}d), \chi b(\overline{c}c) - \gamma_n (d\overline{d}b) = \\
= ((n(c) - \chi \gamma_n n(d))\overline{a}, (\chi n(c) - \gamma_n n(d))b) = 0.
\]
The norm condition (*)

Remark 9

If $n(c) = n(d) = 0$ in Lemma 8 then we can take any $\chi \in \mathbb{R}$. Otherwise, we obtain immediately

$$
\begin{cases}
(n(c))^2 = (n(d))^2 \neq 0; \\
\chi = \gamma_n \frac{n(c)}{n(d)} = \gamma_n \frac{n(d)}{n(c)} = \pm 1.
\end{cases}
$$

Condition (*) is satisfied automatically if $A_{n+1}$ is an algebra of the main sequence or if $A_{n+1}$ is a Cayley-Dickson split-algebra, since in this case $n(c) = n(d)$. The values of $\chi$ are equal to $-1$ and $1$, respectively. However, condition (*) is not true in general.
Extending condition (\(*\))

**Lemma 10 (Moreno, 2006)**

If \(x\) alternates with \(y\) in \(A_n\) then \(n(xy) = n(yx) = n(x)n(y)\).

**Remark 11**

Let \((c, d)\) satisfy condition \((\ast)\) and \((n(a))^2 + (n(b))^2 \neq 0\). Assume without loss of generality that \(n(b) \neq 0\). By Lemma 10, we have

\[
 n(a)n(d) = n(da) = n(-b\bar{c}) = n(b)n(\bar{c}) = n(b)n(c).
\]

Then \(
 \frac{n(a)}{n(b)} = \frac{n(c)}{n(d)} = \chi.
\)

Moreover, Lemma 10 implies that

\[
 \frac{n(ac)}{n(-\chi da)} = \frac{n(ac)}{n(da)} = \frac{n(a)n(c)}{n(a)n(d)} = \frac{n(c)}{n(d)} = \chi.
\]

Hence \((a, b)\) and \((\bar{ac}, -\chi da)\) also satisfy condition \((\ast)\).
A hexagon in $\Gamma_Z(\mathcal{A}_{n+1})$

**Lemma 12**

The elements $ac$, $da$ alternate strongly with $a$, $b$, $c$, $d$.

Hence we may successively apply Lemma 8 to obtain the following corollary.

**Corollary 13**

Let $(a, b)$ and $(c, d)$ satisfy condition $(*).$ Then there exists the following 6-cycle in $\Gamma_Z(\mathcal{A}_{n+1})$ which we call a hexagon:

$$(a, b) \rightarrow (c, d) \rightarrow (ac, -\chi da) \rightarrow (a, -b) \rightarrow \rightarrow (c, -d) \rightarrow (ac, \chi da) \rightarrow (a, b).$$
Properties of zero divisors in $\mathcal{M}_{n+1}$

We now consider the case when $A_{n+1}$ is an algebra of the main sequence, that is, $A_{n+1} = \mathcal{M}_{n+1}$.

**Lemma 14 (Moreno, 1998)**

- Let $x, y \in \mathcal{M}_{n+1}$. Then $xy = 0 \iff yx = 0$.
- If $x \in Z(\mathcal{M}_{n+1})$ then $x$ is doubly pure.
- Let $x = (x_1, x_2), y = (y_1, y_2), \tilde{y} = (-y_2, y_1) \in \mathcal{M}_{n+1}$. Then $xy = 0 \iff x\tilde{y} = 0$.

The next corollary follows immediately from this lemma.

**Corollary 15**

$\Gamma_Z(\mathcal{M}_{n+1})$ can be obtained from $\Gamma_O(\mathcal{M}_{n+1})$ by replacing every undirected edge with a pair of directed edges.
In the case of the algebras of the main sequence Corollary 13 attains the following form.

**Corollary 16**

Let \( a, b \in \mathcal{M}_n \) alternate strongly with \( c, d \in \mathcal{M}_n \), \((a, b)(c, d) = 0\) in \( \mathcal{M}_{n+1} \). Then there exists the following 6-cycle in \( \Gamma_0(\mathcal{M}_{n+1}) \):

\[
(a, b) \leftrightarrow (c, d) \leftrightarrow (ac, ad) \leftrightarrow (a, -b) \leftrightarrow (c, -d) \leftrightarrow (ac, -ad) \leftrightarrow (a, b).
\]

By using Lemma 14 this hexagon can be extended to a double hexagon depicted in Figure 1.
A double hexagon in $\Gamma_0(M_{n+1})$

Figure 1: A double hexagon.
Theorem 17

The multiplication table of the vertices of a double hexagon is given by Table 1, where

\[ f_0 = (e_0, 0) = e_0, \quad \tilde{f}_0 = (0, e_0), \]
\[ f_1 = (a, b), \quad f_4 = (ac, -ad), \quad f_7 = (0, ab), \]
\[ f_2 = (c, -d), \quad f_5 = (c, d), \quad f_8 = (0, dc), \]
\[ f_3 = (ac, ad), \quad f_6 = (a, -b), \quad f_9 = (0, (ac)(ad)), \]
\[ \tilde{f}_1 = (-b, a), \quad \tilde{f}_4 = (ad, ac), \quad \tilde{f}_7 = (-ab, 0), \]
\[ \tilde{f}_2 = (d, c), \quad \tilde{f}_5 = (-d, c), \quad \tilde{f}_8 = (-dc, 0), \]
\[ \tilde{f}_3 = (-ad, ac), \quad \tilde{f}_6 = (b, a), \quad \tilde{f}_9 = (-(ac)(ad), 0). \]
### Multiplication table of the vertices of a double hexagon

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**Table 1:** Multiplication table of the vertices of a double hexagon.
Open questions

- What happens if an element is doubly alternative (its both components are alternative in the previous algebra) but does not satisfy condition (*)?
- What other important properties do the doubly alternative elements which satisfy condition (*) possess? Presently, we have explicit formulae for their annihilators, centralizers, and orthogonalizers.
- How are orthogonality and commutativity graphs of an arbitrary real Cayley-Dickson algebra related?
- Under which conditions are two elements of a real Cayley-Dickson algebras O-equivalent (or C-equivalent), that is, have the same orthogonalizer (or centralizer)?


Thank you for your attention!