# Degree Deviation Measure of Graphs

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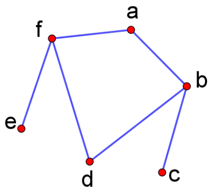
## Degree Deviation Measure of Graphs

- $G$ is a simple, undirected and finite graphs.
- $V(G)$ and $E(G)$ are the vertex set and edge set of $G$, respectively.
- For $v \in V(G)$, $N(v, G)$ is the set of all vertices adjacent to $v$, and the number of vertices in $N(v, G)$ is denoted by $\deg_G(u)$ and called the degree of $v$ in $G$.
- $\Delta = \Delta(G)$ and $\delta = \delta(G)$ are the maximum and minimum degrees of elements in $V(G)$, respectively.
- For integer $i$, $1 \leq i \leq \Delta$, $n_i(G)$ is the number of vertices of degree $i$ in $G$.
- For integer $n$, $H(n)$ is the set of all connected $n-$vertex graphs.

# Degree Deviation Measure of Graphs

## Example

Let $G$ be the following graph. Then $V(G) = \{a, b, c, d, e, f\}$, $E(G) = \{ab, af, bc, bd, fd, fe\}$, $N(a, G) = \{b, f\}$, $deg_G(a) = 2$, $\Delta(G) = 3$, $\delta(G) = 1$ and $n_1 = n_2 = n_3 = 2$.

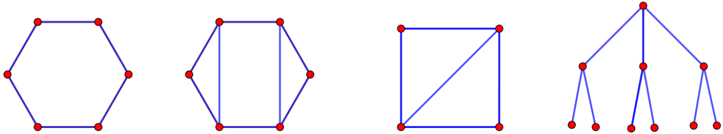


## Degree Deviation Measure of Graphs

- A graph in which all vertices have the same degree is said to be regular graph. Otherwise, the graph is said to be an irregular graph.
- An irregular graph $G$ is said to be bidegreed, if the vertices of $G$ have the degree $\Delta(G)$ or $\delta(G)$.
- An $n$ vertex bidegreed graph $G$ is called a balanced bidegreed graph if $n$ is an even integer and $n_{\Delta(G)} = n_{\delta(G)} = n/2$.
- For a graph $G$ with $n$ vertices and $m$ edges the average degree is equal to $2\mathrm{m}/\mathrm{n}$.
- The cyclomatic number of a connected graph with $n$ vertices and $m$ edges is defined as $m - n + 1$. A tree is a graph with cyclomatic number zero, and for natural number $k$, a $k$-cyclic graph is a graph with cyclomatic number $k$.

# Degree Deviation Measure of Graphs

## Example



Regular
Unicyclic
Average degree=2

Bidegreed
Tricyclic
Average degree=8/3

Balanced bidegreed
Bicyclic
Average degree=5/2

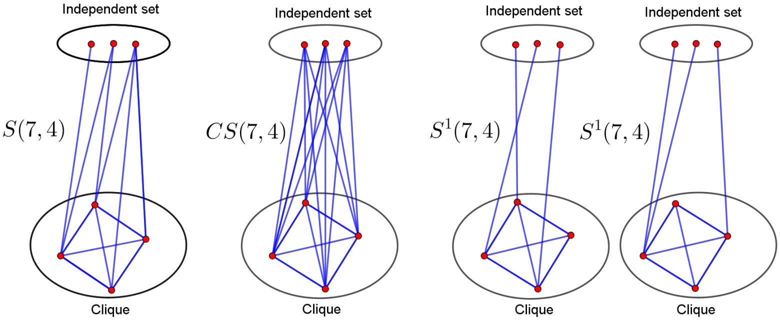
Tree
Average degree=9/5

## Degree Deviation Measure of Graphs

- A clique is a subset of vertices of a graph such that every two distinct vertices in the clique are adjacent, and an independent set in $G$ is a subset of vertices no two of which are adjacent.
- A split graph is a graph in which the vertices can be partitioned into a clique and an independent set. For integers $k$, $0 \leq k \leq n$, a graph $S(n, k)$ is a split graph with a clique of order $k$ and an independent set of order $n - k$.
- A complete split graph, $CS(n, k)$, is a split graph such that each vertex of the clique is adjacent to each vertex of the independent set.
- A starlike split graph, $S^1(n, k)$, is a split graph such that all vertices in its independent set are of degree one.

# Degree Deviation Measure of Graphs

## Example



## Degree Deviation Measure of Graphs

Suppose $\alpha$ is a function from the set of all graphs into non-negative real numbers. If the condition "$\alpha(G) = 0$ if and only if $G$ is regular" is satisfied, then the function $\alpha$ is called a measure of irregularity of graphs. The degree deviation, $S(G)$, and the degree variance, $Var(G)$, of a graph $G$ belong to the family of most popular measures of irregularity of graphs. The degree variance, $Var(G)$, introduced by Bell is formulated as

$$Var(G) = \frac{1}{n} \sum_{v \in V(G)} \left( deg_G(v) - \frac{2m}{n} \right)^2.$$

F. K. Bell, A note on the irregularity of a graph, *Linear Algebra Appl.* **161** (1992) 45-54.

## Degree Deviation Measure of Graphs

The degree deviation, $S(G)$, was introduced by Nikiforov, and for a graph $G$ is defined by

$$S(G) = \sum_{v \in V(G)} \left| deg_G(v) - \frac{2m}{n} \right|.$$

He also obtained the following results:

$$\frac{S(G)^2}{n^2} \leq Var(G) \leq S(G) \quad \text{and} \quad \Omega(G) = \frac{Var(G)}{S(G)} \leq 1$$

V. Nikiforov, Eigenvalues and degree deviation in graphs, *Linear Algebra Appl.* **414** (2006) 347-360.

## Degree Deviation Measure of Graphs

de Oliveira et al. conjectured that

**Conjecture (de Oliveira et al.)**

*Let $H(n)$ be the set of all connected graphs with $n$ vertices. Then $\max_{G\in H(n)} S(G) = S(CS(n,k))$, where*

$$k = \begin{cases} \frac{n}{3} & 3 \mid n \\ \frac{n-1}{3} & 3 \mid n-1 \\ \frac{n-2}{3} \, textand \, \frac{n+1}{3} & 3 \mid n-2 \end{cases}.$$

This talk aims to report a positive answer to above conjecture.

J. A. de Oliveira, C. S. Oliveira, C. Justel and N. M. Maia de Abreu, Measures of irregularity of graphs, *Pesq. Oper.* **33** (3) (2013) 383–398.

## Degree Deviation Measure of Graphs

### Definition

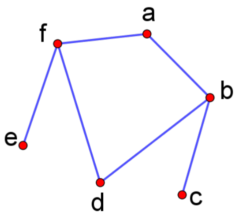
Let $G$ be a graph with $n$ vertices and $m$ edges. Define:

$$
\begin{aligned}
V^{lower}(G) &= \{v \in V(G) \mid deg_G(v) \leq \frac{2m}{n}\}, \\
V^{upper}(G) &= \{v \in V(G) \mid deg_G(v) > \frac{2m}{n}\}, \\
E^{lower}(G) &= \{uv \in E(G) \mid \{u, v\} \subseteq V^{lower}(G)\}, \\
\overline{E}^{upper}(G) &= \{\{u, v\} \subseteq V^{upper}(G) \mid uv \notin E(G)\}.
\end{aligned}
$$

## Degree Deviation Measure of Graphs

### Example

Let $G$ be the following graph. Then $\frac{2m}{n} = 2$, $V^{lower}(G) = \{a, c, d, e\}$, $V^{upper}(G) = \{b, f\}$, $E^{lower}(G) = \{\}$ and $E^{upper}(G) = \{\{f, b\}\}$.



## Degree Deviation Measure of Graphs

The well-known result of Euler, $\sum_{v \in V(G)} deg_G(v) = 2|E(G)|$, implies the following lemma:

### Lemma

- *$V^{lower}(G) = V(G)$ if and only if $G$ is a regular graph.*
- *$V^{upper} \neq V(G)$.*

## Degree Deviation Measure of Graphs

### Corollary

*Let G be an irregular graph. Then* $V^{lower}(G) \neq \emptyset$ *and* $V^{upper}(G) \neq \emptyset$.

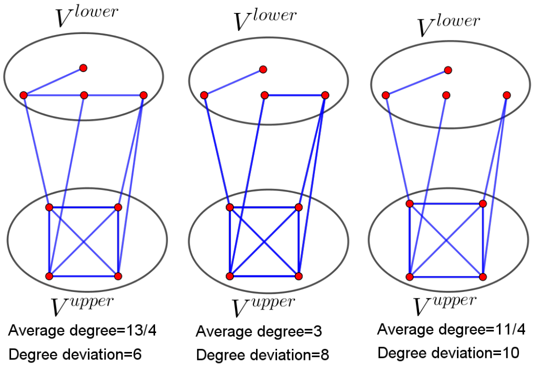
# Degree Deviation Measure of Graphs

## Lemma

*Let $G$ be a connected $n-$vertex irregular graph, $n \geq 3$, and $e = uv \in E^{lower}(G)$ is not a cut edge of $G$. If $G' = G - e$ then $S(G) < S(G')$.*

# Degree Deviation Measure of Graphs

## Example



# Degree Deviation Measure of Graphs

## Definition

Suppose $G$ is a connected irregular graph with a cut edge $e = uv \in E^{lower}(G)$. It is clear that there exists a vertex $w \in V^{upper}(G)$ such that at least one of the graphs $G - uv + uw$ and $G - uv + vw$ is connected. Without loss of generality, we assume that $G - uv + uw$ is connected. Then the edge $uw$ is called a connectedness factor of $G - e$ with respect to $V^{lower}$ and $V^{upper}$.

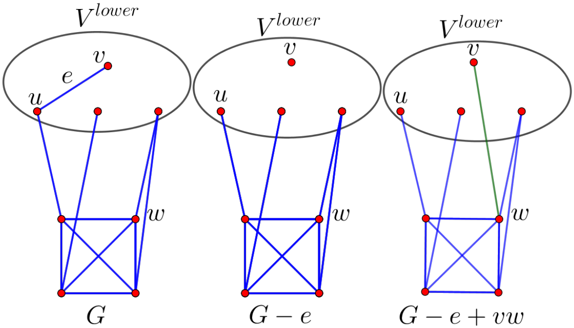
## Degree Deviation Measure of Graphs

### Lemma

*Let $G$ be a connected irregular $n-$vertex graph, $n \geq 3$, and $e = uv \in E^{lower}$ is an cut edge of $G$. If $w \in V^{upper}(G)$ and $uw$ is a connectedness factor of $G - uv$ with respect to $V^{lower}(G)$ and $V^{upper}(G)$, then $S(G) < S(G - uv + uw)$.*

# Degree Deviation Measure of Graphs

## Example




Average degree=11/4
Degree deviation=10

Average degree=11/4
Degree deviation=12

## Degree Deviation Measure of Graphs

By two last lemmas we have the following result:

**Corollary**

*Let $G$ be a graph with maximum degree deviation in the set of all connected graphs with $n$ vertices. Then $V^{lower}(G)$ is an independent set in $G$.*

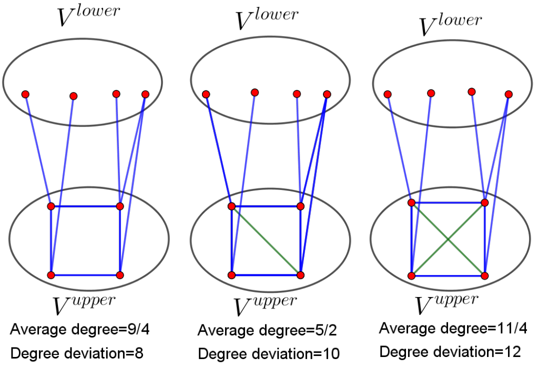
## Degree Deviation Measure of Graphs

### Lemma

*Let $G$ be a connected irregular $n-$vertex graph, $n \geq 3$, and $\{w, z\} \in \overline{E}^{upper}(G)$. If $G^{''} = G + wz$, then $S(G) < S(G^{''})$.*

# Degree Deviation Measure of Graphs

## Example



## Degree Deviation Measure of Graphs

By above discussions, we have the following corollary:

**Corollary**

*Let $G$ be a graph with maximum degree deviation in the set of all connected graphs with $n$ vertices. Then $G$ is a split graph.*

# Degree Deviation Measure of Graphs

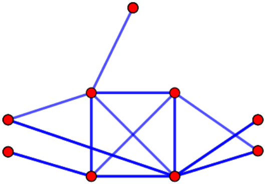
## Lemma

*Suppose $S(n,k)$ is a split graph.*

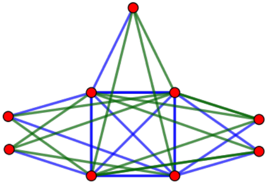
- *If $n-k>k$, then $S(S(n,k)) \leq S(CS(n,k)$, the equality holds if and only if $S(n,k) \cong CS(n,k)$.*
- *If $n-k \leq k$, then $S(S(n,k)) \leq S(S^1(n,k))$.*

## Degree Deviation Measure of Graphs

### Example



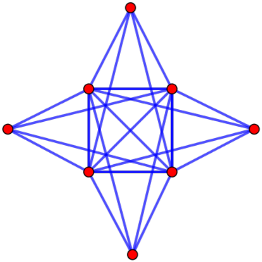
$S = 134/9$



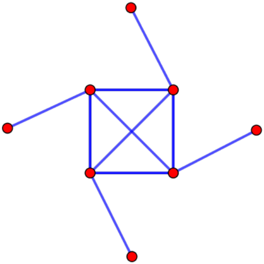
$S = 20$

# Degree Deviation Measure of Graphs

## Example



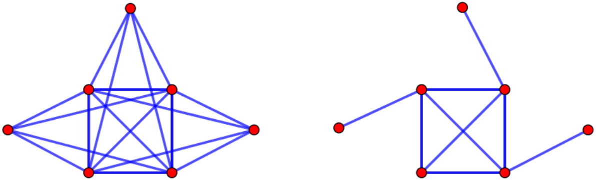
$S = 12$



$S = 12$

# Degree Deviation Measure of Graphs

## Example



$S = 48/7$ $\qquad$ $S = 66/7$

## Degree Deviation Measure of Graphs

By last lemma, we have the following corollary:

### Corollary

*Let $G$ be a graph with maximum degree deviation in the set of all connected graphs with $n$ vertices. Then $G$ is a complete split or starlike split graph.*

## Degree Deviation Measure of Graphs

### Lemma

*Suppose $n$ and $k$ are two positive integers such that $0 \leq k \leq n$.*

- [de Oliveira et al.] *If*

$$k = \begin{cases} \frac{n}{3} & 3 \mid n \\ \frac{n-1}{3} & 3 \mid n-1 \\ \frac{n-2}{3} and \frac{n+1}{3} & 3 \mid n-2 \end{cases},$$

*then $CS(n, k)$ has the maximum degree deviation in the set of all complete split graphs with $n$ vertices.*

## Degree Deviation Measure of Graphs

### Lemma

- *If*

$$k = \begin{cases} \frac{2}{3}n & 3 \mid n \\ \frac{2}{3}(n-1) & 3 \mid n-1 \\ 1 \text{ and } 2 & n = 5 \\ \frac{2}{3}(n+1) & n \neq 5 \text{ and } 3 \mid n-2 \end{cases},$$

*then* $S^1(n,k)$ *has the maximum degree deviation in the set of all starlike split graphs with* $n$ *vertices.*

## Degree Deviation Measure of Graphs

### Lemma

*Let* $CH(n)$ *and* $H(n,1)$ *be the of all complete split graphs and all starlike split graphs on n vertices, respectively. Then*

$$\max_{G \in H(n,1)} S(G) < \max_{G \in CH(n)} S(G).$$

By above discussions, the conjecture of de Oliveira et al. is valid and so we have the following theorem:

**Theorem**

*If*

$$k = \begin{cases} \frac{n}{3} & 3 \mid n \\ \frac{n-1}{3} & 3 \mid n-1 \\ \frac{n-2}{3} \text{ and } \frac{n+1}{3} & 3 \mid n-2 \end{cases},$$

*then $CS(n,k)$ has the maximum degree deviation in the set of all connected graphs with $n$ vertices.*

**THANK YOU**

**FOR YOUR**

**ATTENTION!**

## Choghaznabil Ziggurat, Shush, Khuzestan, Iran, The first Iranian registered building by UNESCO.