Uniqueness and non–uniqueness of prescribed mass NLS ground states on metric graphs

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Nonlinear Schrödinger on metric graphs

Given a metric graph $\mathcal{G}$, consider the NLS energy functional

$$E(u) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\,dx$$

subject to the mass constraint

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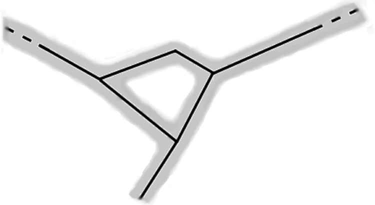
Problem. Are ground states at fixed mass unique?

Physical motivations

Context: propagation of signals along branched structures.



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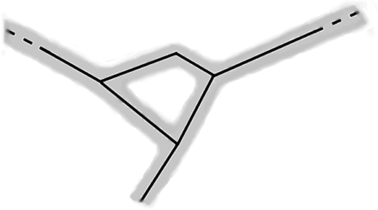

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- Bose–Einstein condensates
- Spectrum of valence electrons in organic molecules
- Nanotechnologies (circuits of quantum wires)
- Spectra of electromagnetic waves in thin dielectrics
- Thin acoustic waveguides
- And more...

Metric graphs

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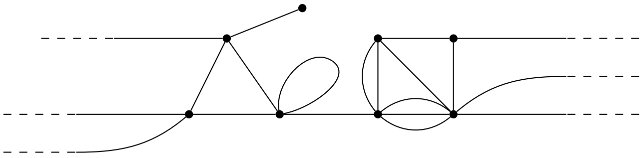

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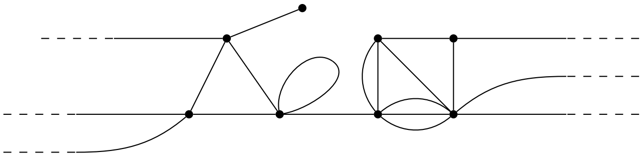

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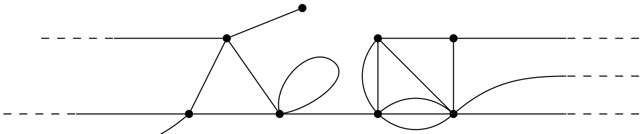

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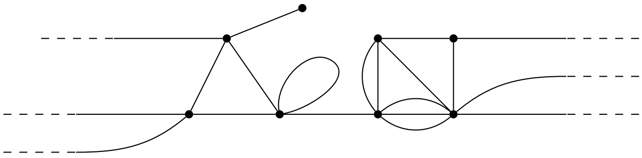

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- With the shortest-path distance, we obtain a metric space $\mathcal{G}$.
- The spaces $L^p(\mathcal{G})$ are defined in the usual way, with Lebesgue measure on every edge.

Metric graphs

The Sobolev space $H^1(\mathcal{G})$ is defined as follows

$$u \in H^1(\mathcal{G}) \iff \begin{cases} u \in H^1(e) & \text{for every edge } e \text{ of } \mathcal{G} \\ u : \mathcal{G} \to \mathbb{R} & \text{is continuous on } \mathcal{G} \end{cases}$$

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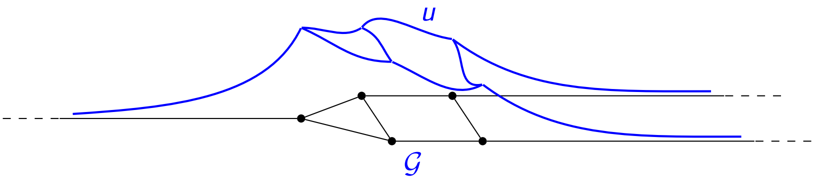
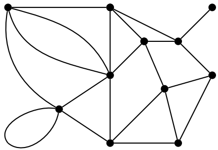
Here is what a typical $H^1(\mathcal{G})$ function looks like:



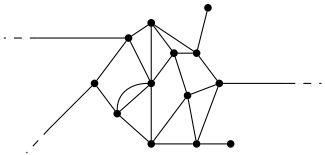
Preliminaries: existence of ground states

Existence/non–existence of ground states has been widely investigated on various graphs.

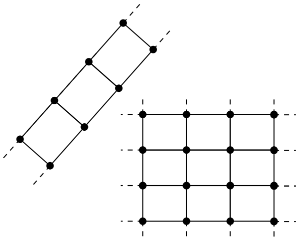
Compact graphs



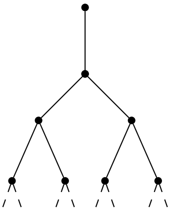
Graphs with half–lines



Periodic graphs



Infinite trees



What about uniqueness?

Let u be a ground state at mass μ. Then there exists $\lambda \in \mathbb{R}$ so that u solves on $\mathcal{G}$ the stationary NLS equation

$$u'' + |u|^{p-2}u = \lambda u. \tag{1}$$

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Main difficulties: nonlinearity, mass constraint, general domains

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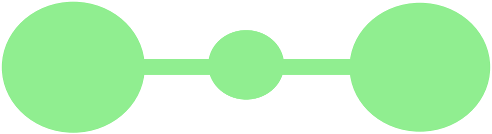

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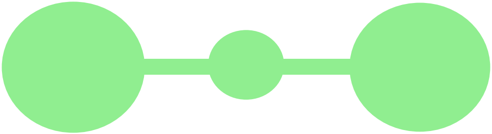

Open conjecture. If Ω is bounded and convex and $2 < p \leq 2^*$, then whenever a positive solution exists, it is unique.

Main results I: uniqueness of λ

Let $\mathcal{G}$ be a given metric graph, $p \in (2, 6]$ and $J \subset \mathbb{R}^+$ be an interval so that ground states at mass μ on $\mathcal{G}$ exist for every $\mu \in J$.

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Theorem (D., Serra, Tilli, Adv. Math. '20)

For all but at most countably many $\mu \in J$:

- *ground states at mass μ share the same $\lambda = \lambda(\mu)$;*
- *$\lambda(\mu)$ is a strictly increasing function of μ;*
- *$\mathcal{E}(\mu) := \inf_{u \in H^1_\mu(\mathcal{G})} E(u)$ is differentiable at μ and*

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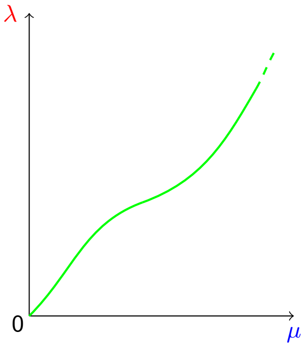
Wide range of applications: subcritical/critical powers, general metric graphs, general domains in $\mathbb{R}^n$.

The dependence of λ on μ may be either



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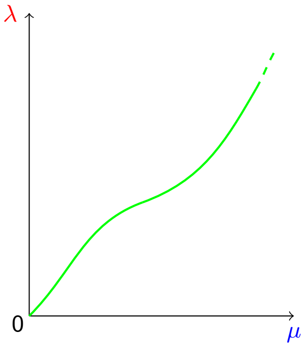

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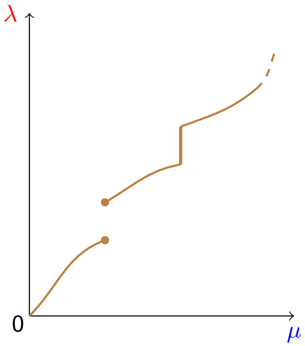

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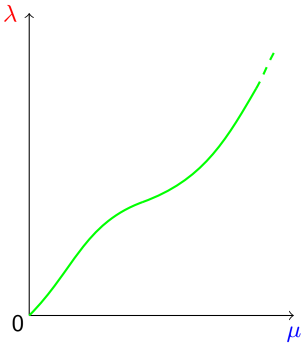

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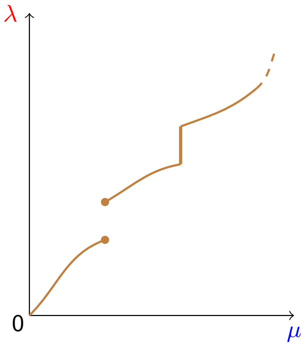

Two natural questions:

- can we prove uniqueness of ground states at masses where λ is unique?
- can we get rid in general of the possible at most countable set of masses where λ may not be unique?

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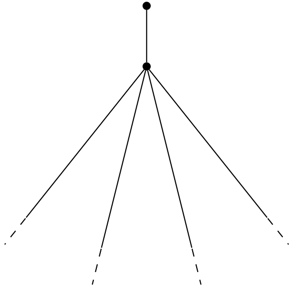

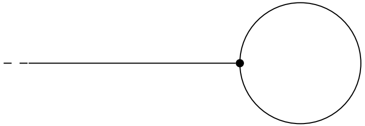

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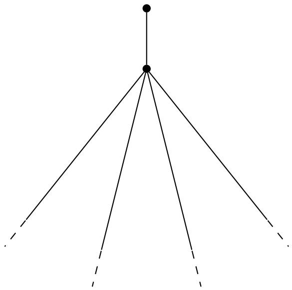

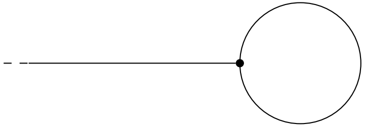

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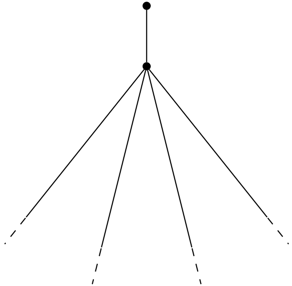

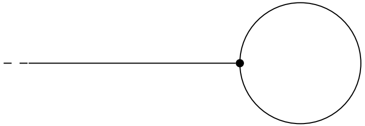

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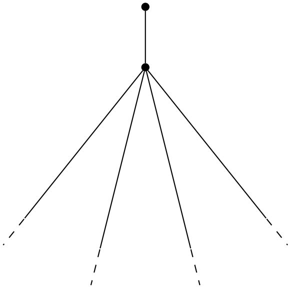

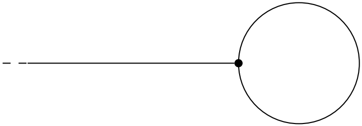

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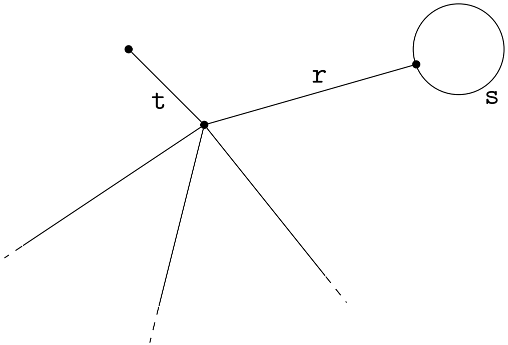
Idea of the proof: given p and μ, we calibrate the graph



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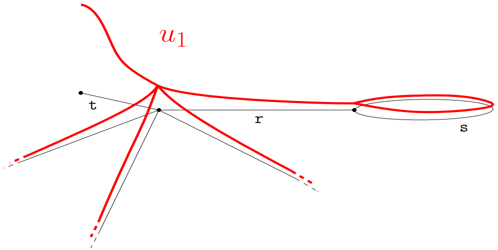

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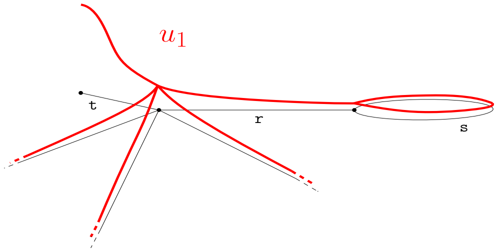
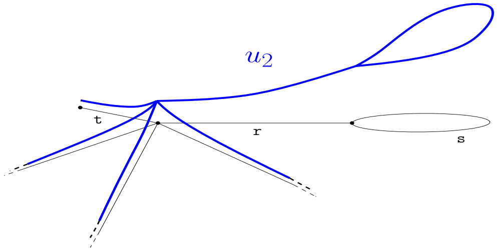
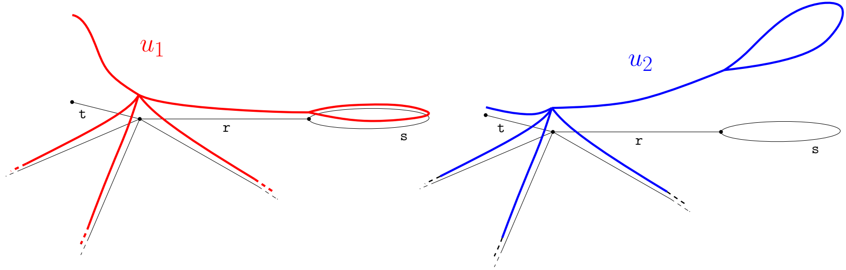


the other concentrates on the self–loop





u_1
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t
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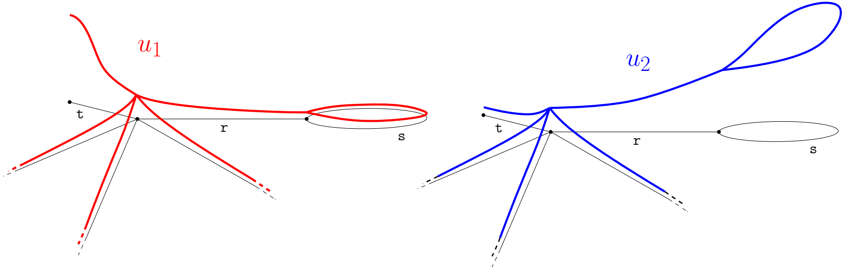

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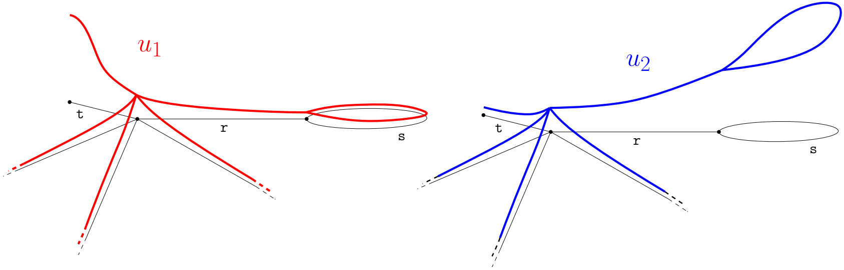

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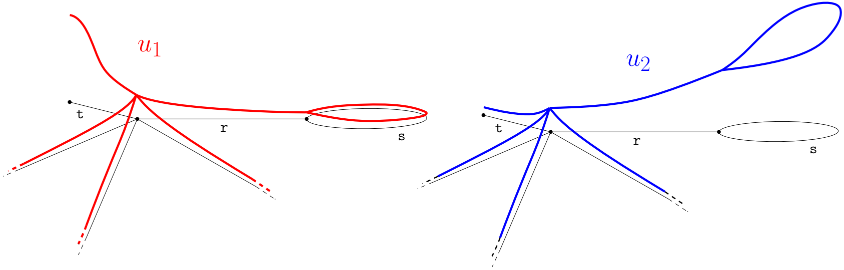

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$$\lambda(u_1) \neq \lambda(u_2)\,.$$

Thank you!