Uniqueness and non–uniqueness of prescribed mass NLS ground states on metric graphs

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Given a metric graph $G$, consider the NLS energy functional

$$E(u) = \frac{1}{2} \int_G |u'|^2 \, dx - \frac{1}{p} \int_G |u|^p \, dx$$

subject to the mass constraint

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**Problem.** Are ground states at fixed mass unique?
Physical motivations

Context: propagation of signals along **branched structures**.

- **Metric graphs** provide one-dimensional approximations for constrained dynamics in which transverse dimensions are negligible compared to longitudinal ones.
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- Bose–Einstein condensates
- Spectrum of valence electrons in organic molecules
- Nanotechnologies (circuits of quantum wires)
- Spectra of electromagnetic waves in thin dielectrics
- Thin acoustic waveguides
- And more...
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- With the shortest-path distance, we obtain a metric space $\mathcal{G}$.
- The spaces $L^p(\mathcal{G})$ are defined in the usual way, with Lebesgue measure on every edge.
The Sobolev space $H^1(G)$ is defined as follows

$$u \in H^1(G) \iff \begin{cases} \ u \in H^1(e) & \text{for every edge } e \text{ of } G \\ \ u : G \to \mathbb{R} & \text{is continuous on } G \end{cases}$$
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Here is what a typical $H^1(G)$ function looks like:
Existence/non–existence of ground states has been widely investigated on various graphs.

- **Compact graphs**
- **Graphs with half–lines**
- **Periodic graphs**
- **Infinite trees**
What about uniqueness?

Let $u$ be a ground state at mass $\mu$. Then there exists $\lambda \in \mathbb{R}$ so that $u$ solves on $G$ the stationary NLS equation

$$u'' + |u|^{p-2}u = \lambda u. \quad (1)$$
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1. do ground states at the same mass \( \mu \) share the same \( \lambda \)?
2. given \( \lambda \), is there a **unique** solution to the stationary NLS \( (1) \)?
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It is a two-level problem:

1. do ground states at the same mass $\mu$ share the same $\lambda$?
2. given $\lambda$, is there a unique solution to the stationary NLS (1)?

Main difficulties: nonlinearity, mass constraint, general domains
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Open conjecture. If $\Omega$ is bounded and convex and $2 < p \leq 2^*$, then whenever a positive solution exists, it is unique.
Main results I: uniqueness of $\lambda$

Let $G$ be a given metric graph, $p \in (2, 6]$ and $J \subset \mathbb{R}^+$ be an interval so that ground states at mass $\mu$ on $G$ exist for every $\mu \in J$.

Theorem (D., Serra, Tilli, Adv. Math. '20)

For all but at most countably many $\mu \in J$:

- ground states at mass $\mu$ share the same $\lambda = \lambda(\mu)$;
- $\lambda(\mu)$ is a strictly increasing function of $\mu$;
- $E(\mu) := \inf_{u \in H^1(G)} E(u)$ is differentiable at $\mu$ and $E'(\mu) = -\frac{1}{2} \lambda \mu$. 

Note.

The proof relies on the minimality of ground states only.

Wide range of applications: subcritical/critical powers, general metric graphs, general domains in $\mathbb{R}^n$. 
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Wide range of applications: subcritical/critical powers, general metric graphs, general domains in $\mathbb{R}^n$. 
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Two natural questions:
- Can we prove uniqueness of ground states at masses where $\lambda$ is unique?
- Can we get rid of the possible at most countable set of masses where $\lambda$ may not be unique?
The dependence of $\lambda$ on $\mu$ may be either $0 \leq \lambda \mu$ or $0 > \lambda \mu$.

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![Graphs showing possible dependences of $\lambda$ on $\mu$.]

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Let $p \in (2, 6]$ and $G$ be either a graph with a pendant and $N$ half-lines or a tadpole.
Main results II: uniqueness of ground states

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Let $p \in (2, 6]$ and $G$ be either a graph with a pendant and $N$ half–lines or a tadpole. Then, for all but at most countably many masses, the ground state at fixed mass is unique.
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- ODE methods to prove uniqueness of positive solutions to
  
  \[ u'' + |u|^{p-2}u = \lambda u. \]
Theorem (D., Serra, Tilli, Adv. Math. '20)

Let $p \in (2, 6)$. For every $\mu > 0$ there exist a graph $G$ that admits two ground states at mass $\mu$ with different $\lambda$. 
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Let \( p \in (2, 6) \). For every \( \mu > 0 \) there exist a graph \( G \) that admits two ground states at mass \( \mu \) with different \( \lambda \).

Our theorem on the uniqueness of \( \lambda \) is sharp: a priori, the at most countable set of masses where it may fail cannot be removed.
Main results III: non-uniqueness


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Our theorem on the uniqueness of $\lambda$ is sharp: a priori, the at most countable set of masses where it may fail cannot be removed.

**Idea of the proof:** given $p$ and $\mu$, we calibrate the graph.
For every $r$ large enough, we construct two critical points of $E$: 
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the other concentrates on the self–loop
There exist large enough sufficiently close to a given value so that both $u_1$ and $u_2$ are ground states at mass $\mu$. But $\lambda(u_1) \neq \lambda(u_2)$. 
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- $s$ large enough
- $t$ sufficiently close to a given value $\bar{t}$
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Thank you!