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Dual bases and orthogonal polynomials

Paweł Woźny

Institute of Computer Science
University of Wrocław, POLAND

e-mail: `Pawel.Wozny@cs.uni.wroc.pl`

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Part I. Dual bases: construction and applications

Dual basis: definition

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- We say that functions
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  2. $\left\langle b_i, d_j^{(n)} \right\rangle = \delta_{ij} \qquad (0 \le i, j \le n)$,

  where $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ii} = 1$

Dual bases: applications

- Representation. If $g \in \mathcal{B}_n$ then

$$g = \sum_{k=0}^{n} a_k b_k, \qquad \text{where} \quad a_k := \left\langle g, d_k^{(n)} \right\rangle \qquad (0 \le k \le n)$$

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- Approximation. For a given function f, the element

$$p^* := \sum_{k=0}^{n} c_k b_k, \qquad \text{where} \quad c_k := \left\langle f, d_k^{(n)} \right\rangle \qquad (0 \le k \le n),$$

is the best approximation of f in the set $\mathcal{B}_n$ in the following sense:

$$\|f - p^*\| = \min_{p \in \mathcal{B}_n} \|f - p\| \qquad (\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle})$$

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Remarks.

1. One can find the optimal element *omitting* the orthogonal basis of the space $\mathcal{B}_n$.

2. Skillful use of dual bases often results in less costly algorithms of solving many computational problems.

Dual bases: recent results and applications

- Dual polynomial bases: Goldman (1994).
- Dual Bernstein polynomials: Ciesielski (1987), Jüttler (1998), Lewanowicz and W. (2006–2012), Rababah and Al-Natour (2007), Chudy and W. (2018, 2021).
- Dual Bézier-Said-Wang basis: Zhang *et al.* (2010).
- Computing roots of polynomials: Barton and Jüttler (2007), Liu *et al.* (2009).
- Degree reduction or merging of Bézier curves and surfaces: Gospodarczyk, Lewanowicz, W. (2009–2017).
- Polynomial approximation of rational Bézier curves and surfaces: Lewanowicz, Keller, W. (2012–2017).
- Construction of dual bases: W. (2013, 2014).
- Dual basis functions in subspaces: Kersey (2013).
- Method of solving boundary value problems: Gospodarczyk, W. (2017).
- Numerical solving of fractional partial differential equations: Jani *et al.* (2017, 2018)

Construction of dual basis: first method

- Let us write the dual function $d_i^{(n)}$ as a linear combination of $b_0, b_1, \ldots, b_n$:

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- One can easily prove that the ith column of the following matrix:

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contains the coefficients

$$\left[a_{i0}^{(n)}, a_{i1}^{(n)}, \ldots, a_{in}^{(n)}\right]^T \qquad (0 \le i \le n)$$

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contains the coefficients $\left[a_{i0}^{(n)}, a_{i1}^{(n)}, \ldots, a_{in}^{(n)}\right]^T \qquad (0 \le i \le n)$.

- Cost: $O(n^3)$

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Theorem. Let α_{ki} be the coefficients in

$$P_k = \sum_{i=0}^{n} \alpha_{ki} b_i \qquad (k = 0, 1, \ldots, n)$$

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Theorem. Let α_{ki} be the coefficients in

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Then the dual basis functions have the following form:

$$d_i^{(n)} = \sum_{k=0}^{n} \alpha_{ki} P_k \qquad (i = 0, 1, \ldots, n)$$

Construction of dual basis: third method

Disadvantages of the methods:

- we have to invert a full matrix (the first method),
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This method does not require:

- the inverting of a full matrix,
- the knowledge of the orthonormal basis

Construction of dual basis: third method – idea

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- Let $k < n$, and $B_k := \{b_0, b_1, \dots, b_k\}$

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- It is possible to find the dual basis

$$D_{k+1} := \left\{ d_0^{(k+1)}, d_1^{(k+1)}, \ldots, d_{k+1}^{(k+1)} \right\}$$

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- There are quite simple connections between elements of the dual bases D_k and D_{k+1}

Construction of dual basis: third method – remarks

- Using this method, we can construct all the dual bases

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- The optimal element g^* is obtained in terms of basis functions $b_0, b_1, \ldots, b_n$. It often results in less costly algorithms of solving many computational problems.

Part II. Dual Bernstein and orthogonal polynomials

Dual Bernstein bases: classical case

- Basis functions ⇒ classical Bernstein basis polynomials (Bernstein, 1912):

$$B_i^n(x) = \binom{n}{i} x^i (1-x)^{n-i} \qquad (0 \le i \le n)$$

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- More on dual Bernstein polynomials and their relations with shifted Jacobi and Hahn (or dual Hahn) polynomials in a few minutes. . .

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- Basis functions ⇒ discrete Bernstein basis polynomials (Neamtu, 1991; Sablonnière, 1992):

$$b_i^n(x;N) = \frac{1}{(-N)_n}\binom{n}{i}(-x)_i(x-N)_{n-i} \qquad (0 \le i \le n \le N;\ N \in \mathbb{N})$$

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$$d_i^n(x;\alpha,\beta,N) = \sum_{k=0}^{n} V_{Nk}^{(\alpha,\beta)}\, Q_k(i;\alpha,\beta,n)\, Q_k(x;\alpha,\beta,N)$$

Dual Bernstein bases: q-case

- Basis functions ⇒ generalized Bernstein basis polynomials (Phillips, $\omega = 0$, 1996; Lewanowicz, W., 2004):

$$B^n_k(x;\,\omega|q) = \frac{1}{(\omega q;q)_n}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k(\omega q/x;q)_k(x;q)_{n-k} \qquad (0 \le k \le n)$$

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- Generalized dual Bernstein polynomials are closely related to big q-Jacobi and q-Hahn polynomials $Q_k(x;a,b,N|q)$ (or dual q-Hahn polynomials).

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$$B_k^n(x;\, \omega|q) = \frac{1}{(\omega q; q)_n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (\omega q/x; q)_k (x; q)_{n-k} \qquad (0 \le k \le n).$$

- Inner product ⇒ big q-Jacobi polynomials $P_k(x;\, a, b, \omega|q)$.

- Dual basis ⇒ generalized dual Bernstein polynomials (Lewanowicz, W., 2006):

$$D_k^n(x; a, b, \omega|q) \in \Pi_n \qquad (0 \le k \le n).$$

- Generalized dual Bernstein polynomials are closely related to big q-Jacobi and q-Hahn polynomials $Q_k(x;\, a, b, N|q)$ (or dual q-Hahn polynomials). For example:

$$D_i^n(x; a, b, \omega|q) = \sum_{k=0}^{n} \qquad\qquad P_k(x;\, a, b, \omega|q)$$

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$$D_i^n(x;a,b,\omega|q) = \sum_{k=0}^{n} V_k^{(a,b,\omega)}(q)\, Q_k\left(q^{i-n};a,b,n|q\right) P_k(x;a,b,\omega|q)$$

Dual Bernstein basis polynomials

- Bernstein basis polynomials of degree n

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$$D_0^n(x;\alpha,\beta),\ D_1^n(x;\alpha,\beta), \ldots, D_n^n(x;\alpha,\beta) \in \Pi_n$$

defined so that

$$\langle D_i^n, B_j^n \rangle_J = \delta_{ij} \qquad (i,j = 0,1,\ldots,n),$$

where

$$\langle f, g \rangle_J := \int_0^1 (1-x)^\alpha x^\beta f(x)\, g(x)\, dx \qquad (\alpha, \beta > -1)$$

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where

$$\langle f, g \rangle_J := \int_0^1 (1-x)^\alpha x^\beta f(x)\, g(x)\, dx \qquad (\alpha, \beta > -1).$$

- Shifted Jacobi polynomials $R_k^{(\alpha,\beta)}(x)$ are orthogonal wrt the inner product $\langle f, g \rangle_J$, i.e.,

$$\left\langle R_k^{(\alpha,\beta)},\, R_l^{(\alpha,\beta)} \right\rangle_J = \delta_{kl} h_k \qquad (k, l = 0, 1, \ldots;\ h_k > 0)$$

Dual Bernstein basis polynomials: important results

- Ciesielski, 1987 ($\alpha = \beta = 0$): definition, recurrence relation.
- Jüttler, 1998 ($\alpha = \beta = 0$): Bernstein-Bézier representation.
- Lewanowicz, W., 2006–2017 ($\alpha, \beta > -1$): recurrence relation, orthogonal expansion, "short" representation, constrained dual Bernstein polynomials, Bézier form, ...
- Rababah, Al-Natour, 2007: extended Jüttler's results to the case $\alpha, \beta > -1$.
- Chudy, W., from 2018 ($\alpha, \beta > -1$): differential-recurrence relations, differential equation, new recurrence relations (for dual Bernstein polynomials of the same degree), ...

Dual Bernstein basis polynomials: explicit formulas

- Recurrence relation

$$D_i^{n+1}(x;\alpha,\beta) = \left(1 - \frac{i}{n+1}\right) D_i^n(x;\alpha,\beta) + \frac{i}{n+1} D_{i-1}^n(x;\alpha,\beta) + \vartheta_i^n R_{n+1}^{(\alpha,\beta)}(x),$$

where ϑ_i^n – simple coefficient

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- Orthogonal expansion

$$D_i^n(x;\alpha,\beta) = \sum_{k=0}^{n} C_k^{(\alpha,\beta)}\, Q_k(i;\beta,\alpha,n)\, R_k^{(\alpha,\beta)}(x),$$

where $Q_k(i;\beta,\alpha,n)$ are Hahn orthogonal polynomials, and $C_k^{(\alpha,\beta)}$ – simple coefficient

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$$D_i^n(x;\alpha,\beta) = \sum_{k=0}^{n} C_k^{(\alpha,\beta)}\, Q_k(i;\beta,\alpha,n)\, R_k^{(\alpha,\beta)}(x).$$

- Short representations

$$D_i^n(x;\alpha,\beta) = A_{ni}^{(\alpha,\beta)} \sum_{k=0}^{i} \frac{(-i)_k}{(-n)_k} R_{n-k}^{(\alpha,\beta+k+1)}(x),$$

$$D_{n-i}^n(x;\alpha,\beta) = B_{ni}^{(\alpha,\beta)} \sum_{k=0}^{i} (-1)^k \frac{(-i)_k}{(-n)_k} R_{n-k}^{(\alpha+k+1,\beta)}(x),$$

where $A_{ni}^{(\alpha,\beta)}$, $B_{ni}^{(\alpha,\beta)}$ – simple coefficients

Dual Bernstein basis polynomials: newer results

- Representation in the basis $(1-x)^j$ $(0 \le j \le n)$

$$D_i^n(x;\alpha,\beta) = \sum_{j=0}^{n} E_{nij}^{(\alpha,\beta)}\, {}_3F_2\left(\begin{matrix} j-n,\ -i,\ 1 \\ -n,\ -n-\alpha \end{matrix}\middle|\ 1\right) \cdot (1-x)^j \qquad (0 \le i \le n),$$

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where $E_{nij}^{(\alpha,\beta)}$ – simple coefficient.

- $F(i,j) := {}_3F_2\left(\left.\begin{matrix} j-n, \ -i, \ 1 \\ -n, \ -n-\alpha \end{matrix}\right| 1\right) \quad (i,j = 0,1,\ldots,n)$

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- The quantities $F(i,j)$ satisfy recurrence relation of the form

$$(i-n)(n-i+\alpha)F(i+1,j) - (i+1)(n+j-i+\alpha+1)F(i,j) = \\ = -(n+1)(n+\alpha+1) \quad (0 \le i,j \le n)$$

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- **Proof**. Zeilberger's algorithm

Dual Bernstein basis polynomials: newer results (cont.)

- Differential-recurrence relations

$$\Big((1-x)\mathbf{D} - (n-i+\alpha+1)\mathbf{I}\Big)D_i^n(x;\alpha,\beta) =$$
$$= \frac{(i-n)(i+\beta+1)}{i+1}D_{i+1}^n(x;\alpha,\beta) - U_{ni}^{(\alpha,\beta)}\frac{n+\alpha+1}{i+1}R_n^{(\alpha,\beta+1)}(x),$$

where $0 \le i \le n$, $\mathbf{D} := \dfrac{\mathrm{d}}{\mathrm{d}x}$, $\mathbf{I}$ – identity operator, a $U_{ni}^{(\alpha,\beta)}$ – simple coefficient

Dual Bernstein basis polynomials: newer results (cont.)

- Differential-recurrence relations

$$\Big((1-x)\mathbf{D}-(n-i+\alpha+1)\mathbf{I}\Big)D_i^n(x;\alpha,\beta)=$$
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$$\Big(x\mathbf{D}+(i+\beta+1)\mathbf{I}\Big)D_i^n(x;\alpha,\beta)=$$
$$=\frac{i(n-i+\alpha+1)}{n-i+1}D_{i-1}^n(x;\alpha,\beta)+U_{ni}^{(\alpha,\beta)}\frac{n+\beta+1}{n-i+1}R_n^{(\alpha+1,\beta)}(x),$$

where $0\le i\le n$, $\mathbf{D}:=\frac{d}{dx}$, $\mathbf{I}$ – identity operator, a $U_{ni}^{(\alpha,\beta)}$ – simple coefficient

Dual Bernstein basis polynomials: newer results (cont.)

- Differential-recurrence relations

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$$\Big(x\mathbf{D}+(i+\beta+1)\mathbf{I}\Big)D_i^n(x;\alpha,\beta)=$$
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where $0\le i\le n$, $\mathbf{D}:=\frac{d}{dx}$, $\mathbf{I}$ – identity operator, a $U_{ni}^{(\alpha,\beta)}$ – simple coefficient.

- **Proof**. Representation of D_i^n in the basis $(1-x)^j$ + recurrence for $F(i,j)$ + symmetry:

$$D_i^n(x;\alpha,\beta)=D_{n-i}^n(1-x;\beta,\alpha)$$

Dual Bernstein basis polynomials: newer results (cont.)

- Third differential-recurrence relation

$$\left(x(x-1)\mathbf{D}^2 + \tfrac{1}{2}(\alpha-\beta+(\sigma+1)(2x-1))\mathbf{D}\right)D_i^n(x;\alpha,\beta) =$$
$$= (i-n)(i+\beta+1)D_{i+1}^n(x;\alpha,\beta) + i(i-\alpha-n-1)D_{i-1}^n(x;\alpha,\beta) -$$
$$-(i(i-\alpha-n-1)+(i-n)(i+\beta+1))D_i^n(x;\alpha,\beta),$$

where $0, 1, \ldots, n$

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  where $0, 1, \ldots, n$.

- **Proof**. Representation in the basis $R_k^{(\alpha,\beta)}(x)$ + differential equation for $R_k^{(\alpha,\beta)}(x)$ + difference equation for Hahn orthogonal polynomials $Q_k(x;a,b;N)$

Dual Bernstein basis polynomials: newer results (cont.)

- Fourth-order differential equation

$$\sum_{j=0}^{4} w_j(x)\mathbf{D}^j\, D_i^n(x;\alpha,\beta) = 0 \qquad (0 \le i \le n),$$

where $w_j(x)$ is a polynomial of degree j $(0 \le j \le 4)$

Dual Bernstein basis polynomials: newer results (cont.)

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- **Proof**. It follows from new differential-recurrence relations

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The *newer* results

$\Downarrow$

- F. Chudy, P. W., *Differential-recurrence properties of dual Bernstein polynomials*, Applied Mathematics and Computation 338 (2018), 537–543

Dual Bernstein basis polynomials: newest results

- First-order non-homogeneous recurrence relation

$$(x-1)(i+1)D_i^n(x;\alpha,\beta)+x(n-i)D_{i+1}^n(x;\alpha,\beta)=$$
$$C_{ni}^{(\alpha,\beta)}\Big((n-i)(n+\alpha+1)xR_n^{(\alpha,\beta+1)}(x)+$$
$$+(i+1)(n+\beta+1)(1-x)R_n^{(\alpha+1,\beta)}(x)\Big),$$

where $C_{ni}^{(\alpha,\beta)}$ – simple coefficient

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where $C_{ni}^{(\alpha,\beta)}$ – simple coefficient.

- **Proof**. Induction on n + properties of shifted Jacobi polynomials

Dual Bernstein basis polynomials: newest results (cont.)

- Third-order recurrence relation

$$\sum_{j=0}^{3} v_j(n, i; x; \alpha, \beta) D^n_{i+j}(x; \alpha, \beta) = 0 \qquad (0 \le i \le n-3),$$

where $v_j(n, i; x; \alpha, \beta)$ – simple cubic polynomial in i (and linear polynomial in x) $(0 \le j \le 3)$

Dual Bernstein basis polynomials: newest results (cont.)

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- **Proof**. It follows from new first-order non-homogeneous recurrence relation

Dual Bernstein basis polynomials: newest results (cont.)

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- F. Chudy, P. W., *Fast and accurate evaluation of dual Bernstein polynomials*, Numerical Algorithms 87 (2021), 1001–1015

Dual Bernstein basis polynomials: newest results (cont.)

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- **Problem.** Let us fix numbers: $n \in \mathbb{N}$, $x \in [-1, 1]$ and $\alpha, \beta > -1$. Consider the problem of computing the values

$$D_i^n(x; \alpha, \beta)$$

for all $i = 0, 1, \ldots, n$

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- **Solution.**

First-order non-homogeneous recurrence relation $\Rightarrow$ $O(n)$

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Previous results $\Rightarrow$ $O(n^2)$

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- Tests: veeery good numerical properties :-)

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- (Classical) Dual Bernstein polynomials:
  - three differential-recurrence relations,
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- Classical, discrete and q families of dual Bernstein polynomials have nice connections with classical, discrete and q families of orthogonal polynomials.
- (Classical) Dual Bernstein polynomials:
  - three differential-recurrence relations,
  - fourth-order differential equation,
  - first-order non-homogeneous recurrence relation,
  - third-order recurrence relation.
- For a given x, α, β, the values of all $n + 1$ (classical) dual Bernstein polynomials $D_k^n(x; \alpha, \beta)$ $(0 \leq k \leq n)$ can be computed with the complexity $O(n)$