Decay of solutions to integrodifferential equations

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Integrodifferential equation (viscoelasticity)

$$u_{tt} - \Delta u + f(x,u) + \int_0^t k(s)\Delta u(t-s)\,\mathrm{d}s = g \quad \text{in } \mathbb{R}_+ \times \Omega \qquad \text{(IDE)}$$

- $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$,
- $\Omega \subset \mathbb{R}^N$ bounded domain,
- $u = 0$ on $\mathbb{R}_+ \times \partial\Omega$

GOAL

- $\lim_{t\to+\infty} u(t) = \varphi$ and
- $\|u(t) - \varphi\| \le Ct^{-\xi}$ for all $t \ge 0$

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Correspondence with wave equation

$$u_{tt} - \Delta u + f(x,u) + \int_0^t k(s)\Delta u(t-s)\mathrm{d}s = g$$

Abstract wave equation

$$u_{tt} + cu_t + \Phi'(u) = 0 \tag{WE}$$

$$u_{tt} + \Phi'(u) + \int_0^t k(s)Au(t-s)\mathrm{d}s = g \tag{GE}$$

Abstract wave equation

$$u_{tt} + cu_t + \Phi'(u) = 0$$

Define Lyapunov functional $\mathcal{E}(u) = \frac{1}{2}\|u_t\|^2 + \Phi(u)$

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(u(t)) &= u_t u_{tt} + \Phi'(u)u_t \\ &= u_t(u_{tt} + \Phi'(u)) \\ &= -cu_t^2 \leq 0\end{aligned}$$

$$\mathcal{E}(u(t)) \to \mathcal{E}_\infty \qquad \overset{?}{\Rightarrow} \qquad u(t) \to \varphi$$

$$u_t(t) \to 0, \qquad \Phi'(u(t)) \to 0$$

- Φ convex, then $u(t) \to \varphi$
- Φ has many (continuum) critical points, then . . . we need more

For ugly Φ ...

...we need more

- Let $V \hookrightarrow H \hookrightarrow V^*$ be Hilbert spaces. A functional $\Phi : M \subset V \to \mathbb{R}$ of class C^1 satisfies the Łojasiewicz–Simon inequality if $\forall \varphi \; \exists \theta \in (0, \frac{1}{2}]$, C, $\varepsilon > 0$ s.t.

$$|\Phi(u) - \Phi(\varphi)|^{1-\theta} \leq C\|\Phi'(u)\|_{V^*} \qquad \forall u \in B(\varphi, \varepsilon) \qquad \text{(LI)}$$

  θ is called **the Łojasiewicz exponent.**

- better Lyapunov functional: $\mathcal{E}(u) = \frac{1}{2}\|u_t\|^2 + \Phi(u) + \varepsilon\|u_t\|\Phi(u)$ ($\frac{d}{dt}\mathcal{E}(u(t)) \leq 0$ is not enough)

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Back to (IDE)

$$\mathcal{E}(u) = \frac{1}{2}\|u_t\|^2 + \Phi(u) + \int_t^\infty \langle g(s), u_t(s)\rangle\, ds - \frac{1}{2}\int_0^t k(s)\, ds \|A^{1/2}u(t)\|^2 + \frac{1}{2}\int_0^t k(t-s)\|A^{1/2}u(s) - A^{1/2}u(t)\|^2\, ds$$

$$\frac{d}{dt}\mathcal{E}(u(t)) \leq 0$$

$$\tilde{\mathcal{E}}(u) = \mathcal{E}(u) + \varepsilon(\dots)$$

Assumptions on (GE)

$$\ddot{u} + \Phi'(u) - \int_0^t k(s) A u(t-s) \mathrm{d}s = g \tag{GE}$$

Assumptions on A and Φ: (A)

- $V \subset H \subset V^*$ Hilbert spaces
- $\Phi : V \to \mathbb{R}$ of class C^2
- A linear, self-adjoint, positive definite with $V \subset D(A^{1/2})$
- $\Phi_A(u) = \Phi(u) - \frac{\mu}{2}\|A^{1/2}u\|^2$ satisfies (LI)

where $\mu = \int_0^{+\infty} k$

Assumptions on k and g: (K) (G)

- $k \in C^1(\mathbb{R}_+)$
- there exist $p \in (2, +\infty]$ and $c_k \geq 0$ s.t. $k' \leq -c_k k^{1+\frac{1}{p}}$ for all $t \geq 0$
- $g \in L^1(\mathbb{R}_+, H) \cap L^2(\mathbb{R}_+, H)$
- either there exists $q \in (1, +\infty)$ s.t. $\int_t^\infty \|g(s)\|^2 ds \leq c_g(1+t)^{-2q+1}$,
- or $\int_t^\infty \|g(s)\|^2 ds \leq c_g e^{-2\gamma t}$ ('case $q = +\infty$')

Assumptions on k imply

- $k(t) \leq Ct^{-p}$ if $p < +\infty$
- $k(t) \leq Ce^{-ct}$ if $p = +\infty$

Assumptions on g are satisfied e.g. if

- $\|g(t)\| \leq C(1+t)^{-q}$ (case $q < +\infty$)
- $\|g(t)\| \leq Ce^{-\gamma t}$ (case $q = +\infty$)

Theorem

Let (A), (k), (g) hold and let $u : \mathbb{R}_+ \to V$ *be a strong solution to (GE) such that* $\{(u(t), \dot{u}(t) : \ t \in \mathbb{R}_+\} \subset V \times H$ *is relatively compact.*
Then there exists $\varphi \in V$ *such that*

$$\lim_{t \to +\infty} \|\dot{u}(t)\|_H + \|u(t) - \varphi\|_V = 0.$$

Moreover, for every $\varepsilon > 0$ *there exists* $C > 0$ *such that*

$$\|u(t) - \varphi\|_H \le ct^{-\xi}, \quad \textit{for all } t \ge 0$$

where $\xi = \min\{\frac{\theta}{1-2\theta}, q - \frac{1}{2}, \frac{p}{2} - 1 - \varepsilon\}$.

Remark

If $p = q = +\infty$ *and* $\theta = \frac{1}{2}$, *we have*

$$\|u(t) - \varphi\|_H \le Ce^{-ct} \quad \textit{for all } t \ge 0$$

for appropriate c, $C > 0$.

Assumptions on (IDE)

$$u_{tt} - \Delta u + f(x,u) + \int_0^t k(s)\Delta u(t-s)\mathrm{d}s = g \quad \text{in } \mathbb{R}_+ \times \Omega \qquad \text{(IDE)}$$

Assumptions on non-linearity f: (F)

- $f = f(x,u) \in C^2(\mathbb{R}_+ \times \Omega)$
- analytic in u uniformly w.r.t. to $x \in \Omega$ and u in bounded subsets of $\mathbb{R}$
- $f(\cdot, 0) \in L^\infty(\Omega)$ and there exists ρ, $\alpha > 0$, $(N-2)\alpha < N$ s.t.

$$\left| \frac{\partial f}{\partial u}(x,u) \right| \le C(1 + |u|^\alpha)$$

Then (Haraux, Jendoubi '99)

$$E_\mu(u) = \frac{1-\mu}{2}\|\nabla u\|_2 + \int_\Omega F(x,u)dx$$

satisfies (LI)

Theorem

Let (F), (k), (g) hold with $\int_0^\infty k = \mu < 1$ and let $u : \mathbb{R}_+ \to V$ be a strong solution to (IDE) such that $\{(u(t), \dot{u}(t) : \ t \in \mathbb{R}_+\} \subset V \times H$ is relatively compact.
Then there exists $\varphi \in V$ such that

$$\lim_{t \to +\infty} \|\dot{u}(t)\|_H + \|u(t) - \varphi\|_V = 0.$$

Moreover, for every $\varepsilon > 0$ there exists $C > 0$ such that

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Remark

If $p = q = +\infty$ and $\theta = \frac{1}{2}$, we have

$$\|u(t) - \varphi\|_H \leq Ce^{-ct} \quad \textit{for all } t \geq 0$$

Comments on Theorems

Operators satisfying (LI): (Chill '03, Ben Hassen, Haraux '11)

- Laplacian with non-linearity f and Neumann or Robin b. c.
- D. or N. Laplacian with non-analytic non-linearities
- fourth order $\Delta^2 - \lambda_1 u + c_1(u_+)^p - c_2(u_-)^q$
- $E(u) = \frac{1}{2}M(a(u,u)) + \int_\Omega c(x)u^2$, where a is positive bilinear form and M a real function
- $\operatorname{div} a(x, \nabla u) - c(x)u$

Precompactness of trajectory

Decay estimates are not optimal (Haraux '11, Abdelli and Haraux '14, B. '18)

(LI) for equations with dynamic boundary conditions (Chill, fašangová, Prüss '06, Wu '10, '12)

THANK YOU FOR YOUR ATTENTION!