Hardy–Littlewood–Sobolev inequality for $p = 1$

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The Riesz potential

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$$\mathrm{I}_\alpha[f](x) = \int\limits_{\mathbb{R}^d} |\xi|^{-\alpha} \hat{f}(\xi) e^{2\pi i \langle x, \xi \rangle}\, d\xi, \quad x \in \mathbb{R}^d,\ f \in L_1(\mathbb{R}^d).$$

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This operator appears frequently in the study of PDE (it might be though of as "inverse differentiation"), or in geometric measure theory (to measure dimension of sets), or in many other places.

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Theorem (Hardy–Littlewood–Sobolev inequality)

The operator I_α *is bounded as an* L_p *to* L_q *mapping if and only if* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}$ *and* $1 < p < q < \infty$.

  

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Theorem (Classical Sobolev embedding)

If $\frac{1}{p} - \frac{1}{q} = \frac{1}{d}$ *and* $1 < p < q < \infty$, *then*

$$\|f\|_{L_q} \lesssim \|\nabla f\|_{L_p}, \qquad f \in C_0^\infty(\mathbb{R}^d).$$

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Using the Calderón–Zygmund theory (or rather the Mikhlin mulitplier theorem), this may be restated as

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Recall that HLS fails for $p = 1$ when tested against a delta measure. In a sense, the Gagliardo–Nirenberg inequality says that the vectorial expression ∇f cannot concentrate as well as delta measures do.

So it is natural to ask: "For what spaces (classes of functions) W does the inequality

$$\| \mathrm{I}_\alpha[g]\|_{L_{\frac{d}{d-\alpha}}} \lesssim \|g\|_{L_1}, \qquad g \in W,$$

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Theorem (Hardy and Littlewood, 1927)

The inequality

$$\|I_{\frac12} f\|_{L_2(\mathbb{R})} \lesssim \|f\|_{L_1(\mathbb{R})}, \qquad \hat{f}(\xi) = 0, \quad \xi < 0,$$

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Theorem (Hardy and Littlewood, 1927)

The inequality

$$\|I_{\frac{1}{2}} f\|_{L_2(\mathbb{R})} \lesssim \|f\|_{L_1(\mathbb{R})}, \qquad \hat{f}(\xi) = 0, \quad \xi < 0,$$

holds true.

In modern terms this may be stated as $\mathrm{I}_\alpha \colon H_1 \to L_{\frac{d}{d-\alpha}}$, where H_1 is either complex or the real Hardy class (Stein–Weiss inequality).

More examples: Bourgain–Brezis inequalities

Yet another example was found about twenty years ago by Bourgain and Brezis:

$$\| \mathrm{I}_1[g]\|_{L_{\frac{d}{d-1}}} \lesssim \|g\|_{L_1}, \quad \mathrm{div} g = 0.$$

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Theorem (Van Schaftingen, 2011)

The inequality

$$\|\,\mathrm{I}_1[g]\|_{L_{\frac{d}{d-1}}(\mathbb{R}^d)} \lesssim \|g\|_{L_1(\mathbb{R}^d)}, \quad Ag = 0,$$

where A is a homogeneous elliptic differential operator holds true if and only if the space $* - \mathrm{clos}_{\mathrm{Meas}}(\{Ag = 0 \mid g \in C_0^\infty(\mathbb{R}^d)\})$ does not contain vectorial delta measures.

My result

Theorem

Let $\mathcal{W}$ be a closed linear subspace of $\mathcal{S}'(\mathbb{R}^d,\mathbb{R}^\ell)$ that is invariant under translations and dilations, let $\alpha \in (0,d)$. The constant in the inequality

$$\|\,\mathrm{I}_\alpha[f]\|_{L_{\frac{d}{d-\alpha}}} \lesssim \|f\|_{L_1}, \qquad f \in \mathcal{W},$$

is uniform with respect to all $f \in \mathcal{W}$, for which the right hand side is finite, if and only if $\mathcal{W}$ does not contain the charges $a \otimes \delta_0$, $a \in \mathbb{R}^\ell \setminus \{0\}$.

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For those analysts who are interested, we may replace the space $L_{\frac{d}{d-\alpha}}$ on the left with a finer space $L_{\frac{d}{d-\alpha},1}$; this solves several open problems going back to Van Schaftingen and even Bourgain–Brezis (here I am glad to mention that Hernandez and Spector solved B.-B. conjecture independently).

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$$\|\mathrm{H}[\mu](\cdot,t)\|_{L_p(\mathbb{R}^d)} \leq t^{-\frac{d(p-1)}{2p}+\delta}\|\mathrm{H}[\mu](\cdot,1)\|_{L_p(\mathbb{R}^d)}.$$

One then has to decompose f into parts where it resembles a measure in

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Thank you!