Some boundary properties of nonlocal minimal surfaces

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Nonlocal minimal surfaces

Energy functional dealing with “pointwise interactions” between a given set and its complement

Main idea: the “surface tension” is the byproduct of long-range interactions

Implications: nonlocal phase transitions and nonlocal capillarity theories

New effects due to the long-range interactions

Contributions from “far-away” can have a significant influence on the local structures of these new objects

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The fractional perimeter functional

Given $s \in (0, 1)$ and a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1,\gamma}$-boundary, the $s$-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in $\Omega$ is defined as

$$
\text{Per}_s(E; \Omega) := L(E \cap \Omega, (CE) \cap \Omega) + L(E \cap \Omega, (CE) \cap (C\Omega)) + L(E \cap (C\Omega), (CE) \cap \Omega),
$$

where $CE = \mathbb{R}^n \setminus E$ denotes the complement of $E$, and $L(A, B)$ denotes the following nonlocal interaction term

$$
L(A, B) := \int_A \int_B \frac{1}{|x - y|^{n+s}} \, dx \, dy \quad \forall \, A, B \subseteq \mathbb{R}^n,
$$

This notion of $s$-perimeter and the corresponding minimization problem were introduced in [Caffarelli-Roquejoffre-Savin, 2010].
1) **Existence theorem:**
   there exists $E$ s-minimizer for $\text{Per}_s$ in $\Omega$ with $E \setminus \Omega = E_0 \setminus \Omega$.

2) **Maximum principle:**
   $E$ s-minimizer and $(\partial E) \setminus \Omega \subset \{|x_n| \leq a\} \Rightarrow \partial E \subset \{|x_n| \leq a\}$.

3) If $\partial E$ is an hyperplane, then $E$ is s-minimizer.

4) If $E$ is s-minimizer in $B_1$, then $\partial E$ is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set $\Sigma$, with Hausdorff dimension less or equal than $n - 2$.

5) If $E$ is s-minimizer and $0 \in \partial E$, then
   \[
   \int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{E^c}(y)}{|y|^{n+s}} dy = 0.
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[Savin-Valdinoci, 2013]:

Regularity of cones in dimension 2.

If $E$ is $s$-minimizer in $B_1$, then $\partial E$ is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set $\Sigma$, with Hausdorff dimension less or equal than $n - 3$. 
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Limit as $s \to 1$


$$(1 - s)\text{Per}_s \to \text{Per}, \quad \text{as } s \nearrow 1$$

(up to normalizing multiplicative constants).

\[\downarrow\]

[Caffarelli-Valdinoci, 2013]:
$s$ close to 1: nonlocal minimal surfaces are as regular as classical minimal surfaces.

(If $E$ is $s$-minimizer in $B_1$, then $\partial E$ is $C^{1,\alpha}$ in $B_{1/2}$ except in a closed set $\Sigma$, with Hausdorff dimension less or equal than $n - 8$.)
Limit as $s \to 1$


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Limit as $s \to 0$

[Maz’ya-Shaposhnikova, 2002] and [Dipierro-Figalli-Palatucci-Valdinoci, 2013]:

If there exists the limit

$$\alpha(E) := \lim_{s \downarrow 0} s \int_{E \cap (CB_1)} \frac{1}{|y|^{n+s}} \, dy,$$

then

$$\lim_{s \downarrow 0} s \text{Per}_s(E, \Omega) = \left( \omega_{n-1} - \alpha(E) \right) \frac{|E \cap \Omega|}{\omega_{n-1}} + \alpha(E) \frac{|\Omega \setminus E|}{\omega_{n-1}}.$$
Stickiness to half-balls

For any $\delta > 0$,

$$K_\delta := \left( B_{1+\delta} \setminus B_1 \right) \cap \{ x_n < 0 \}.$$

We define $E_\delta$ to be the set minimizing the $s$-perimeter among all the sets $E$ such that $E \setminus B_1 = K_\delta$. 
There exists $\delta_o > 0$ such that for any $\delta \in (0, \delta_o]$ we have that

$$E_\delta = K_\delta.$$
Given a large $M > 1$ we consider the $s$-minimal set $E_M$ in $(-1, 1) \times \mathbb{R}$ with datum outside $(-1, 1) \times \mathbb{R}$ given by the jump $J_M := J_M^- \cup J_M^+$, where

$$J_M^- := (-\infty, -1] \times (-\infty, -M)$$

and

$$J_M^+ := [1, +\infty) \times (-\infty, M).$$
Stickiness to the sides of a box

There exist $M_o > 0$ and $C_o \geq C'_o > 0$, depending on $s$, such that if $M \geq M_o$ then

$$[-1, 1) \times [C_o M^{\frac{1+s}{2+s}}, M] \subseteq E^c_M$$

and

$$(-1, 1] \times [-M, -C_o M^{\frac{1+s}{2+s}}] \subseteq E_M.$$

Also, the exponent $\beta := \frac{1+s}{2+s}$ above is optimal.
Stickiness to the sides of a box
We consider a sector in $\mathbb{R}^2$ outside $B_1$, i.e.

$$\Sigma := \{(x, y) \in \mathbb{R}^2 \setminus B_1 \text{ s.t. } x > 0 \text{ and } y > 0\}.$$ 

Let $E_s$ be the $s$-minimizer of the $s$-perimeter among all the sets $E$ such that $E \setminus B_1 = \Sigma$. Then, there exists $s_o > 0$ such that for any $s \in (0, s_o]$ we have that $E_s = \Sigma$. 
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Stickiness as $s \to 0^+$
Instability of the flat fractional minimal surfaces

Fix $\epsilon_0 > 0$ arbitrarily small. Then, there exists $\delta_0 > 0$, possibly depending on $\epsilon_0$, such that for any $\delta \in (0, \delta_0]$ the following statement holds true.

Assume that $F \supset H \cup F_- \cup F_+$, where

$$H := \mathbb{R} \times (-\infty, 0),$$

$$F_- := (-3, -2) \times [0, \delta)$$

and

$$F_+ := (2, 3) \times [0, \delta).$$

Let $E$ be the $s$-minimal set in $(-1, 1) \times \mathbb{R}$ among all the sets that coincide with $F$ outside $(-1, 1) \times \mathbb{R}$.

Then

$$E \supseteq (-1, 1) \times (-\infty, \delta^{\frac{2+\epsilon_0}{1-s}}].$$
Instability of the flat fractional minimal surfaces

\[ \beta := \frac{2 + \epsilon_0}{1 - s} \]
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A useful barrier
(Dis)connectedness of nonlocal minimal surfaces
[Dipierro-Onoue-Valdinoci, 2020]

We consider nonlocal minimal surfaces in a cylinder with prescribed datum given by the complement of a slab.

\[ \Omega := \{ (x', x_n) \text{ with } |x'| < 1 \}. \]

\[ E_0 := \{ (x', x_n) \text{ with } |x'| > M \}. \]
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As in the classical case, when the width of the slab is large the minimizers are disconnected and when the width of the slab is small the minimizers are connected.

Differently from the classical case, when the width of the slab is large the minimizers are not flat discs, and when the width of the slab is small then the minimizers completely adhere to the side of the cylinder.
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There exists $m_0 \in (0, 1)$ such that if $M \in (0, m_0)$, then the minimizer in $\Omega$ coincides with $\Omega$. In particular, it is connected (but it does not look like a catenoid!).
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There exists $M_0 > 1$ such that if $M > M_0$, then the minimizer in $\Omega$ is disconnected.

Differently from the classical case, the minimizer contains

$$B_{cM^{-s}}(0, \ldots, 0, -M) \cup B_{cM^{-s}}(0, \ldots, 0, M),$$

so it is not the complement of a slab. Also (at least in dimension 2) it sticks at the boundary.
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Yin-Yang Theorems

...com'è difficile trovare l'alba dentro l'imbrunire...
Yin-Yang Theorems
[Bucur-Dipierro-Lombardini-Valdinoci, 2020]

There exists $\vartheta > 1$ such that if $E$ is $s$-minimal in $\Omega \subset \mathbb{R}^n$ and $E \cap (\Omega_{\vartheta \text{diam}(\Omega)} \setminus \Omega) = \emptyset$, then

$$E \cap \Omega = \emptyset.$$
Thank you very much for your attention!