# Nonlocal minimal graphs are generically sticky

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Nonlocal Operators and Related Topics – EMS 2021

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# Outline

Nonlocal minimal surfaces

Energy functional dealing with *"pointwise interactions"* between a given set and its complement

Main idea: the "surface tension" is the byproduct of long-range interactions

Implications: nonlocal phase transitions and nonlocal capillarity theories

New effects due to the long-range interactions

Contributions from "far-away" can have a significant influence on the local structures of these new objects

STICKINESS Differently from classical minimal surfaces, the nonlocal minimal surfaces have the **strong tendency to "stick at the boundary"**

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# The fractional perimeter functional

Given $s \in (0,1)$ and a bounded open set $\Omega \subset \mathbb{R}^n$ with $C^{1,\gamma}$-boundary, the $s$-perimeter of a (measurable) set $E \subseteq \mathbb{R}^n$ in $\Omega$ is defined as

$$\begin{aligned}\mathrm{Per}_s(E;\Omega) &:= L(E\cap\Omega,(\mathcal{C}E)\cap\Omega)\\ &\quad + L(E\cap\Omega,(\mathcal{C}E)\cap(\mathcal{C}\Omega)) + L(E\cap(\mathcal{C}\Omega),(\mathcal{C}E)\cap\Omega),\end{aligned}$$

where $\mathcal{C}E = \mathbb{R}^n \setminus E$ denotes the complement of $E$, and $L(A,B)$ denotes the following nonlocal interaction term

$$L(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\,dx\,dy \qquad \forall A,B \subseteq \mathbb{R}^n,$$

This notion of $s$-perimeter and the corresponding minimization problem were introduced in [Caffarelli-Roquejoffre-Savin, 2010].

# Instability of the flat fractional minimal surfaces



Fix $\epsilon_0 > 0$ arbitrarily small. Then, there exists $\delta_0 > 0$, possibly depending on $\epsilon_0$, such that for any $\delta \in (0, \delta_0]$ the following statement holds true.
Assume that $F \supset H \cup F_- \cup F_+$, where

$$H := \mathbb{R} \times (-\infty, 0),$$

$$F_- := (-3, -2) \times [0, \delta)$$

and

$$F_+ := (2, 3) \times [0, \delta).$$

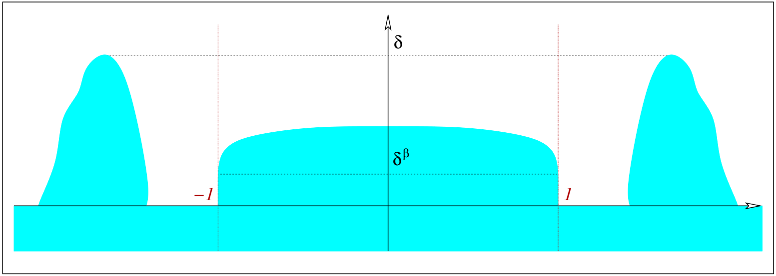
Let $E$ be the $s$-minimal set in $(-1, 1) \times \mathbb{R}$ among all the sets that coincide with $F$ outside $(-1, 1) \times \mathbb{R}$.
Then

$$E \supseteq (-1, 1) \times (-\infty, \delta^{\frac{2+\epsilon_0}{1-s}}].$$

# Instability of the flat fractional minimal surfaces



$\beta := \frac{2+\epsilon_0}{1-s}$



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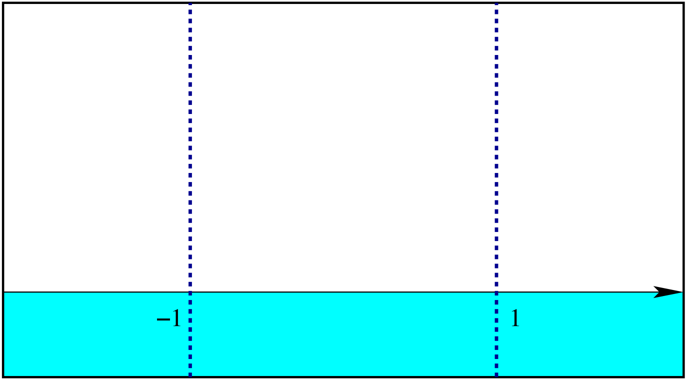

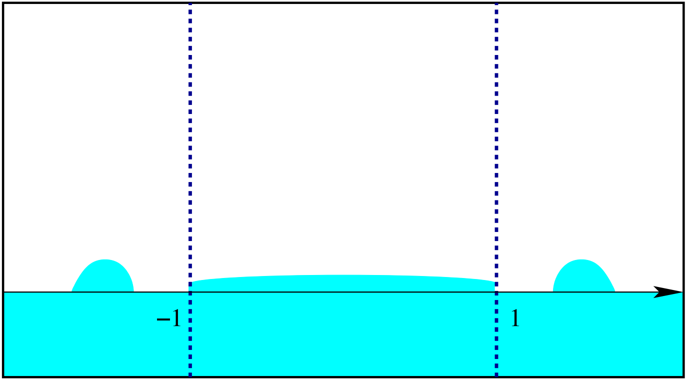

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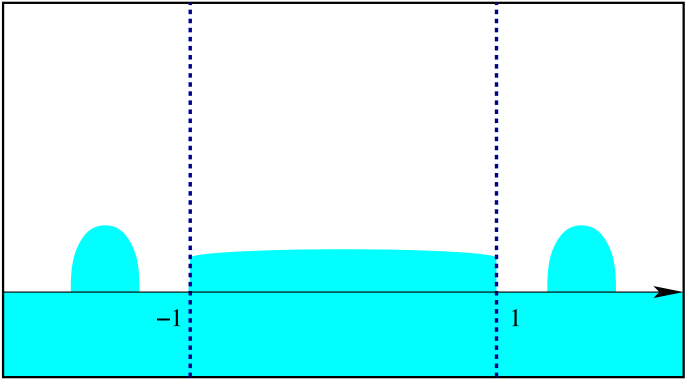
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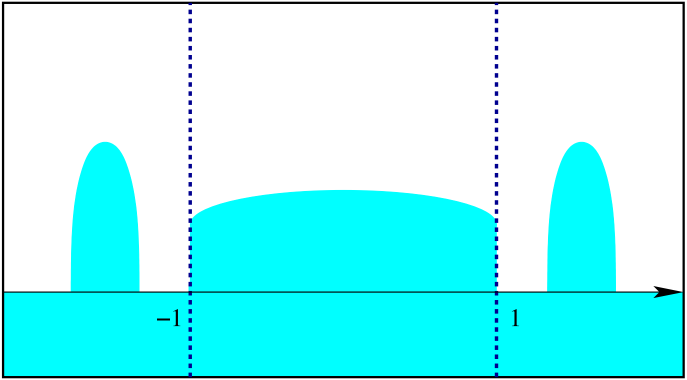

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# Three further questions [Dipierro-Savin-Valdinoci, 2020]



1. How regular are the nonlocal minimal surfaces *coming from inside the domain*?

2. Is the Euler-Lagrange equation satisfied *up to the boundary*?

3. How *typical* is the stickiness phenomenon?

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# Regularity coming from inside



“Continuity implies differentiability”

Consider a nonlocal minimal graph in $(0, 1)$, with a smooth external graph $u_0$.

There is a dichotomy:

- either
$$\lim_{x\nearrow 0} u_0(x) \neq \lim_{x\searrow 0} u(x)$$
and
$$\lim_{x\searrow 0} |u'(x)| = +\infty,$$
- or
$$\lim_{x\nearrow 0} u_0(x) = \lim_{x\searrow 0} u(x)$$
and $u$ is $C^{1,\frac{1+s}{2}}$ at 0.

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# Some remarks

This dichotomy is a purely nonlinear effect, since the boundary behavior of linear equation is of Hölder type [Serra-Ros Oton].

# Some remarks

Stickiness + dichotomy = butterfly effect

An arbitrarily small perturbation of the flat data produce a boundary discontinuity, which entails an infinite derivative at the boundary.

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As a curve, the nonlocal minimal graph turns out to be always $C^{1,\frac{1+s}{2}}$:

it is either the graph of a $C^{1,\frac{1+s}{2}}$-function (when it is continuous at the boundary!), or it is discontinuous and sticks vertically detaching in a $C^{1,\frac{1+s}{2}}$ fashion [Caffarelli-De Silva-Savin] (then the inverse function is a $C^{1,\frac{1+s}{2}}$ function).

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The nonlocal mean curvature can be written in the form

$$\int_{-\infty}^{+\infty} F\left(\frac{u(x+y)-u(x)}{|y|}\right)\frac{dy}{|y|^{1+s}}.$$

And this is a "$C^{1,s}$ operator".

But $\frac{1+s}{2} > s$, therefore we can "pass the equation to the limit"...

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# Boundary Euler-Lagrange equations



If $u$ is a nonlocal minimal graph in $(0,1)$ with smooth datum outside, then

$$\int_{-\infty}^{+\infty} F\left(\frac{u(x+y)-u(x)}{|y|}\right)\frac{dy}{|y|^{1+s}} = 0$$

for all $x \in [0,1]$.

# Stickiness is generic

Let $\varphi \in C_0^\infty([-2,-1],\,[0,1])$, with $\varphi \not\equiv 0$.

Let $u^{(t)}$ be the nonlocal minimal graph in $(0,1)$ with external datum

$$u_0^{(t)} := u_0 + t\varphi.$$

Suppose that

$$\lim_{x\nearrow 0} u_0(x) = \lim_{x\searrow 0} u(x).$$

Then

$$\lim_{x\nearrow 0} u_0^{(t)}(x) < \lim_{x\searrow 0} u^{(t)}(x).$$

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With the Euler-Lagrange equation up to the boundary, we can take any configuration, add an arbitrarily small bump and use the unperturbed configuration as a barrier.

At touching points the additional bump produces an extra-mass violating the Euler-Lagrange equation.

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# Proof of dichotomy

Think about the usual suspects (discontinuous, Lipschitz, Hölder, smooth).

Blow-up.

The “worst” cases to understand are the Hölder and the smooth (the Lipschitz produces non-minimal corners).

The smooth case produces flat objects: use a boundary improvement of flatness (combined with a boundary monotonicity formula) to deduce smoothness of the initial minimizer (for this, use new barrier to go beyond the linear theory!).

The Hölder case produces vertical angles: rule them out by proving that close-to-vertical nonlocal minimal graphs are indeed vertical (for this, slide balls).

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While stickiness in dimension 2 corresponds to a boundary discontinuity, in dimension 3 or higher even more complicated phenomena can arise.

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Let $u$ be $s$-minimal in $(-1,1)\times(0,1)\times\mathbb{R}$ with $u=0$ in $(-2,2)\times\left(-\frac{1}{100},0\right)$.

Consider the trace of $u$

$$f(x) := \lim_{y\searrow 0} u(x,y).$$

Assume that $f(0)=0$. Then, near the origin,

$$|u(x,y)| \le C\left(x^2+y^2\right)^{\frac{3+s}{4}}.$$

In particular, $f'(0)=0$.

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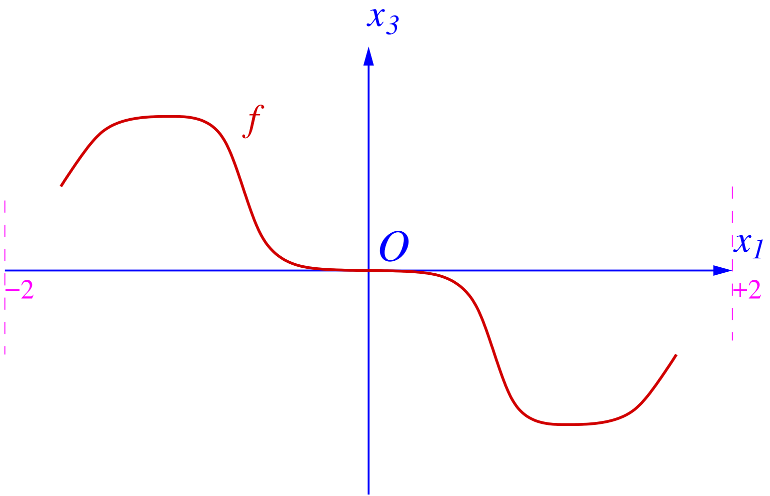
Vanishing of the gradient of the trace at the zero crossing points



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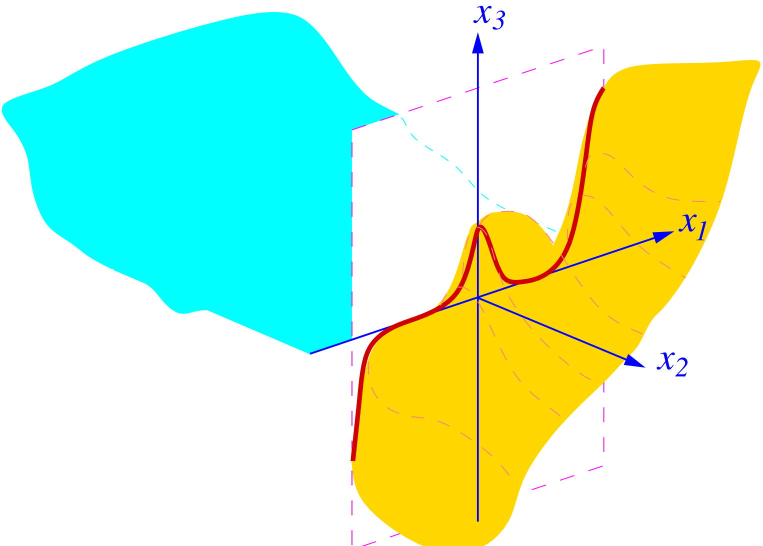

# Stickiness in dimension 3 [Dipierro-Savin-Valdinoci, 2020]

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On the other hand, boundary points with a jump have necessarily a vertical tangency.

Consequently, points with vertical tangency accumulate to zero crossing points possessing horizontal tangency, preventing a differentiable boundary regularity in a neighborhood of horizontal points!

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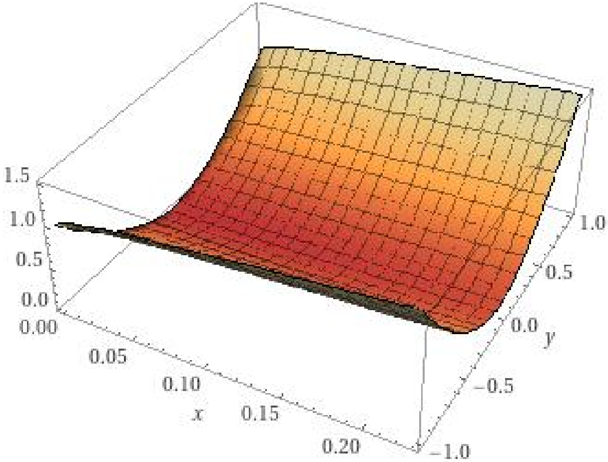
...a bit complicated to plot. Think, for instance, to the function

$$(x^2 + y^2)^{7/8}(1 + x^{4/7}) \qquad \text{with } x \in (0,1), y \in (-1,1).$$



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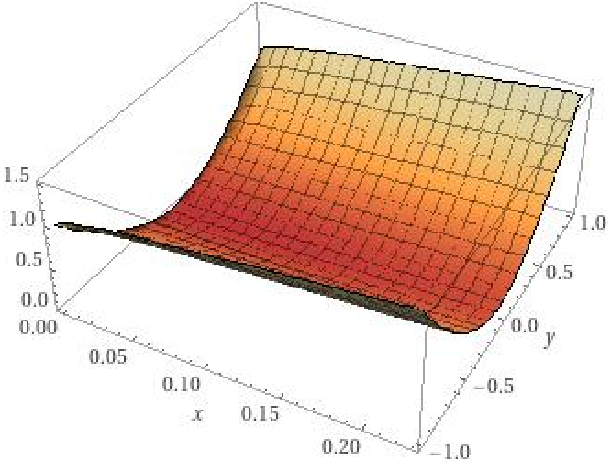

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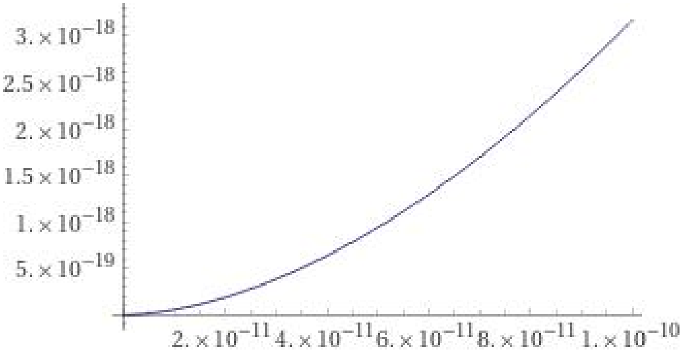


$y = 0$

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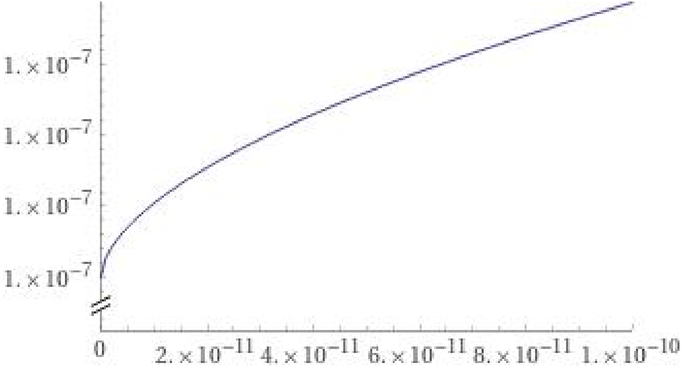


$y = 10^{-4}$

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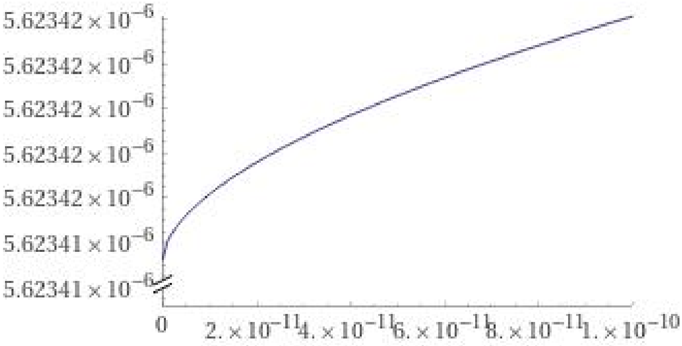


$y = 10^{-3}$

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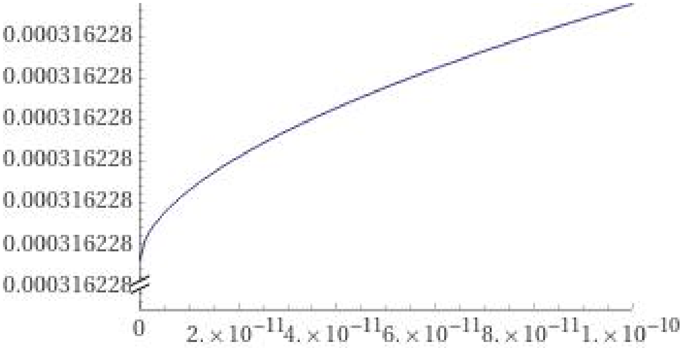


$y = 10^{-2}$

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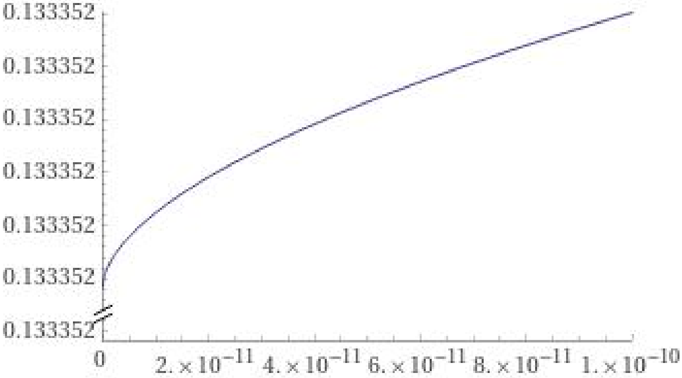


$y = 1$

# Stickiness in dimension 3 [Dipierro-Savin-Valdinoci, 2020]

Pivotal step: if a homogeneous nonlocal minimal graph in $\{x > 0\}$ vanishes in $\{x < 0\}$ and is continuous at the origin, then it necessarily vanishes at all points:

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be an $s$-minimal graph in $\{x > 0\}$, with $u = 0$ in $\{x < 0\}$.

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(Dimension 3 was "easier" because the trace is a function from $\mathbb{R}$ to $\mathbb{R}$, so knowing the derivative at a point, together with the 1-homogeneity, determines already half of the trace; in dimension 4 this only determines the trace along a half-line).

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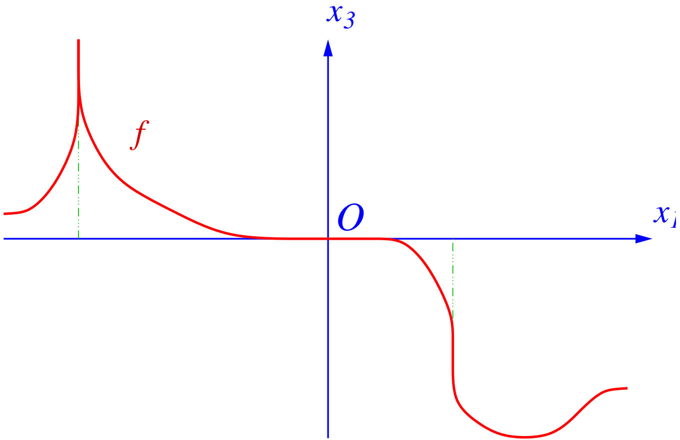


Is it possible to construct examples of nonlocal minimal graphs which are locally flat from outside and whose trace develops vertical tangencies?

# Open problems [Dipierro-Savin-Valdinoci, 2020]



What is the behavior of a nonlocal minimal graph and of its trace at the corners of the domain and in their vicinity?

Can one understand (dis)continuity and tangency properties, possibly also in relation with the convexity or concavity of the corner?

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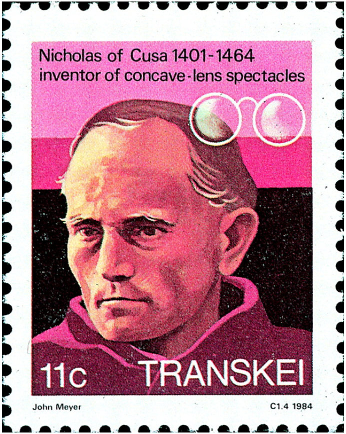


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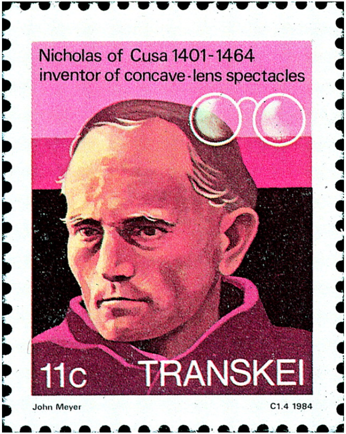


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## How “nonlinear” is the problem?

The linearization of the trace of a nonlocal minimal graph is given by the fractional normal derivative of a fractional Laplace problem.

Indeed, if $u$ is a nonlocal minimal graph, say in $x \in (0,1)$, and it is $\varepsilon$-flat near the origin, then $\frac{u}{\varepsilon}$ (the “vertical rescaling”) tends to a function $\overline{u}$ which is a solution of $(-\Delta)^{\frac{1+s}{2}}\overline{u}(x) = 0$ for $x \in (0,1)$.

By the boundary regularity of linear equation (Serra, Ros-Oton, Grubb, etc.) the first order of $\overline{u}$ is of Hölder type: near the origin $\overline{u} \simeq ax^{\frac{1+s}{2}}$, for some $a \in \mathbb{R}$.

So, one may expect that, near the origin, $u(x) \simeq a\varepsilon x^{\frac{1+s}{2}}$.

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# Flexibility of linear equations [Dipierro-Savin-Valdinoci, 2020]



But this is not the case! The fractional normal derivative of a fractional Laplace problem is not only different than zero in general, but it can be arbitrarily prescribed:

Let $n \geq 2$ and $f \in C(\mathbb{R}^{n-1})$. Then, for every $\delta > 0$ there exist $f_\delta, u_\delta \in C(\mathbb{R}^{n-1})$ such that

$$\begin{cases} \sup_{|x'|\leq 1} |f_\delta(x') - f(x')| \leq \delta, \\ (-\Delta)^\sigma u_\delta = 0 \text{ in } B_1 \cap \{x_n > 0\}, \\ u_\delta = 0 \text{ in } \{x_n < 0\}, \\ \lim_{x_n \searrow 0} \frac{u_\delta(x)}{x_n^\sigma} = f_\delta(x') \text{ for all } |x'| < 1. \end{cases}$$

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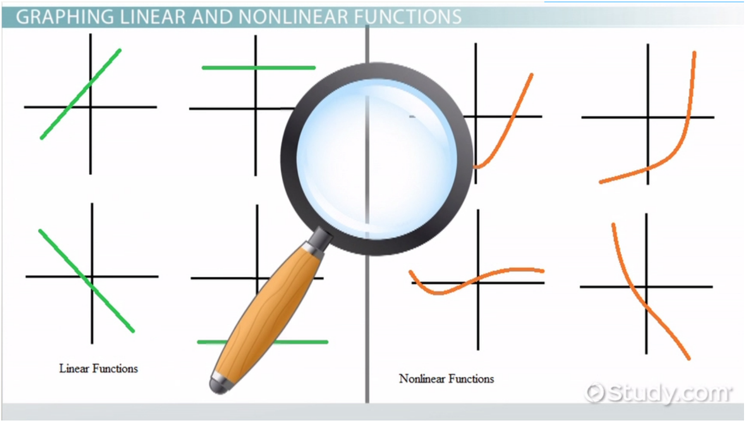


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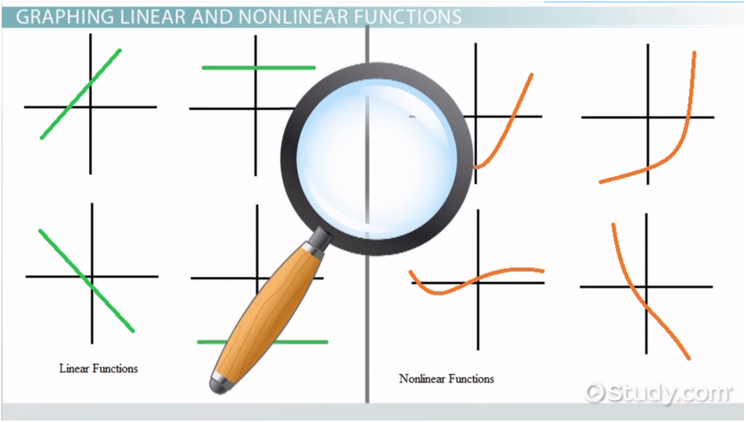


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# Thank you very much for your attention!