Attractors of dissipative homeomorphisms of the infinite surface homeomorphic to a punctured sphere

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joint work with R. Ortega and A. Ruiz-Herrera. 8 ECM, minisymposium Topological methods in dynamical systems
Motivation:

Several results in the literature have provided sufficient conditions to guarantee a simple structure of the attractor. On the other hand, there are not so much results in the opposite direction. Our work was inspired by Fumio Nakajima who gave an interesting result that relates the local behavior of a dynamical system with the complexity of the attractor.

Consider a smooth planar dynamical system $f$ having two distinct fixed points, one of which is an inverse saddle. By an inverse saddle $p$ we understand a fixed point $p$ such that the eigenvalues of $Df(p)$, $\lambda_1$ and $\lambda_2$, satisfy

$$\lambda_1 < -1 < \lambda_2 < 0.$$ 

Nakajima proved that: the existence of an inverse saddle (under some additional assumptions) implies that the attractor is not arcwise connected.
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We will consider self-maps of the infinite surface homeomorphic to a \( s \)-punctured sphere \( M_s \).

**Definition**

We call a homeomorphism of \( M_s \) dissipative if there exists a compact set that attracts uniformly all compact sets. For a dissipative homeomorphism \( h : M_s \to M_s \) we consider the attractor \( \mathcal{A} \subset M_s \), defined as the maximal compact invariant set. This set always exists and is a non-empty continuum. In the giving setting the attractor \( \mathcal{A} \) may also be equivalently defined as the set of all bounded (forward and backward) orbits.
The class of considered maps

Definition

We will call a homeomorphism \( h : M_s \to M_s \) a Levinson homeomorphism if it is dissipative and \( \mathcal{A} \) has an empty interior.

Let us remark that Levinson homeomorphisms contain the important class of dissipative homeomorphisms contracting some Borel measure on \( M_s \).
What is the maximum number of Jordan curves in the attractor of Levinson homeomorphisms on the Cylinder?

**Lemma**

For a Levinson homeomorphisms on the Cylinder the attractor $\mathcal{A}$ contains either none Jordan curve or one Jordan curve that is not contractible.

**Proof.** Let $R(\Gamma_1, \Gamma_2)$ be the union of the bounded components of $C \setminus (\Gamma_1 \cup \Gamma_2)$. It turns out that all orbits in $R(\Gamma_1, \Gamma_2)$ are bounded and thus $R(\Gamma_1, \Gamma_2) \subset \mathcal{A}$, which contradicts the fact that $\mathcal{A}$ has empty interior.
Main result for $s = 2$, i.e. for $M_2$ being a Cylinder

**Theorem**

Let $h$ be a Levinson homeomorphism of a Cylinder. Assume that $h$ is isotopic to identity and has an inverse saddle $p$. Then the attractor $A$ is not arcwise connected.

**Sketch of the proof.**
1. First we prove that there exist another fixed point $q$.
2. Next, we assume contrary to our claim that $A$ is arcwise connected, and consider an arc $\gamma$ joining $p$ and $q$.
3. We consider $h(\gamma)$ and prove that there is a Jordan curve $\Gamma$ in $\gamma \cup h(\gamma)$.
4. Due to the fact that $p$ is an inverse saddle there must be another Jordan curve $h(\Gamma)$, thus we get two Jordan curves and we have the contradiction with Lemma.
From a more applied point of view, the above Theorem offers new dynamical insights on the global attractor in some classical models coming from non-conservative mechanics. This perspective of our results is related to Rogerio Martins’ work who proved that the attractor of the Poincaré map associated with the pendulum equation with friction is not homeomorphic to $S^1$ provided there exists an inverse unstable fixed point. Under slightly more restrictive assumptions our theorem guarantees a stronger property: that the attractor is not arcwise connected.
The counterpart of mentioned above Theorem for the space $M_s$, where $s > 2$.

Let us denote by $\text{Mod}(M_s)$ the mapping class group of $M_s$ i.e. the group of isotopy classes of orientation-preserving homeomorphisms of $M_s$.

It is convenient to interpret topologically $M_s$ as the sphere $S^2$ with punctures. Notice that two homeomorphisms $h_1, h_2$ belong to the same isotopy class if the behavior of their extensions $\tilde{h}_1, \tilde{h}_2$ coincide on the punctures. In other words, $\tilde{h}_1$ permutes the punctures in the same manner as $\tilde{h}_2$.

We denote by $T(h)$ the permutation of the punctures mentioned above.
**Theorem**

Let \( h \in \mathcal{LH}(M_s), \ s > 2 \). Assume that \( h \) has an inverse saddle \( p \) and that there is another fixed point \( q \neq p \) of \( h \). Assume also that \( T(h) \) is a product of disjoint odd cycles. Then the attractor \( A \) is not arcwise connected.

**Remark**

Theorem 5 does not hold if the assumption that \( T(h) \) is a product of odd cycles is dropped.
Counterexample on $M_3$

Let us take $S^2 = \mathbb{R}^2 \cup \{\infty\}$, $z_1 = (1, 0)$, $z_2 = (-1, 0)$, $z_3 = \infty$. We interpret $M_3$ as $\mathbb{R}^2 \setminus \{z_1, z_2\}$. 

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Thank you for your attention!
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