Optimal control problem for a repulsive chemotaxis system

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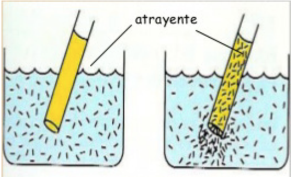
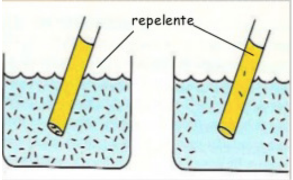
1. Chemotaxis PDE models
   - Chemorepulsion - Production
2. Optimal control problem
   - Well-posedness of State problem.
   - Existence of global minimum
   - Lagrange multipliers
3. Conclusions and Work in progress
   - Conclusions
   - Work in progress

Chemotaxis phenomenum

Chemotaxis is the biological process of the movement of living organisms in response to a chemical stimulus which can be given towards a higher (attractive) or lower (repulsive) concentration of a chemical substance. At the same time, the presence of living organisms can produce or consume chemical substance.





Chemotaxis system [Keller-Segel, 1970-1971]

$$\begin{cases} \partial_t u - \Delta u \pm \nabla \cdot (u \nabla v) = 0 \text{ in } \Omega,\ t > 0, \\ \partial_t v - \Delta v + v = g(u) \text{ in } \Omega,\ t > 0, \\ \frac{\partial v}{\partial \mathrm{n}} = \frac{\partial u}{\partial \mathrm{n}} = 0 \text{ on } \partial\Omega,\ t > 0, \\ v(\mathrm{x}, 0) = v_0(\mathrm{x}) \geq 0,\ u(\mathrm{x}, 0) = u_0(\mathrm{x}) \geq 0 \text{ in } \Omega, \end{cases} \tag{1}$$

- u denotes the cell density,
- v is the chemical concentration,
- The term $\pm u \nabla v$ models the transport of cells by
  - **Chemo-attraction** $(+)$
  - **Chemo-repulsion** $(-)$
- The reaction term $g(u) \geq 0$ models **Production**

Some (general) properties

- Positivity: $u \geq 0$ and $v \geq 0$.
- u conservation: Integrating $(1)_1$ in Ω,

$$\frac{d}{dt}\left(\int_\Omega u\right) = 0, \quad \text{i.e.} \quad \int_\Omega u(t) = \int_\Omega u_0, \quad \forall t > 0.$$

Theorem (Local classical solution and extensibility criteria)

$$\begin{cases} \partial_t u - \Delta u \pm \nabla \cdot (u \nabla v) = f(t,x,u,v) \\ \partial_t v - \Delta v + v = g(t,x,u,v), \end{cases}$$

Hypotheses: $\Omega \subset \mathbb{R}^N$

- *Regularity:* $f, g \in C^1([0,+\infty) \times \overline{\Omega} \times \mathbb{R}^2_+)$
- *Positivity:* $f(t,x,0,v) \geq 0$, $g(t,x,u,0) \geq 0$ *(if* u, $v \geq 0$*)*
- *ICs:* $(u_0, v_0) \in C^0(\overline{\Omega}) \times W^{1,q}(\Omega)$ *for some* $q > N$.

Then

- *Local in time:* $\exists\, T_{max} \in (0,+\infty]$, $\exists!$ u, $v \geq 0$ *a classical solution* u, $v \in C^{1,2}((0, T_{max}) \times \overline{\Omega})$,
- *Extensibility:* *If* $T_{max} < \infty$, *then*
  $\limsup_{t \to T_{max}} \left(\|u(t,\cdot)\|_{L^\infty} + \|v(t,\cdot)\|_{W^{1,q}} \right) \to +\infty$

Chemorepulsion with potential production

$$\begin{cases} \partial_t u - \Delta u - \nabla \cdot (u \nabla v) = 0, \\ \partial_t v - \Delta v + v = u^p, \end{cases} \tag{2}$$

- $p = 1$: linear term
- $1 < p \leq 2$: superlinear term ($p = 2$ quadratic).
- $p > 2$ (Open Problem)

Positivity: $u \geq 0$ and $v \geq 0$.
u conservation: $\int_\Omega u(t) = \int_\Omega u_0$

Energy equality: Chemotaxis and Production cancel

Using test functions $(F'(u) = \log(u)$ or $u^{p-1}, -\frac{1}{p}\Delta v)$:

$$\frac{d}{dt}\mathcal{E}(u,v) + \mathcal{D}(u,v) = 0,$$

where $\mathcal{E}(u,v) = \int_\Omega F(u) + \frac{1}{2p}\int_\Omega |\nabla v|^2$

and $\mathcal{D}(u,v) = \int_\Omega F''(u)|\nabla u|^2 + \frac{1}{p}\int_\Omega(|\nabla v|^2 + |\Delta v|^2)$

$F(u) = u\log(u) - u$ (linear), $F(u) = \frac{1}{p}u^p$ (superlinear)

Energy regularity

$$\left(\sqrt{u\log(u)} \text{ or } u^{p/2},\ \nabla v\right) \in L^\infty L^2 \cap L^2 H^1,$$

Global in time estimate for $\left(\int_\Omega v\right)$:

$$\frac{d}{dt}\left(\int_\Omega v\right) + \int_\Omega v = \int_\Omega u^p \leq C \quad \Rightarrow \int_\Omega v \leq C$$

3D Interpolation regularity

$$u^{p/2} \in L^{10/3}_{t,x} \quad \Rightarrow \quad u^p \in L^{5/3}_{t,x} \quad \Rightarrow \quad u \in L^{5p/3}_{t,x}$$

$$\nabla v \in L^{10/3}_{t,x} \quad \Rightarrow \quad u\,\nabla v \in L^q_{t,x} \quad q = 10p/(3p+6)$$

$$\nabla u = u^{1-p/2}\nabla(u^{p/2}) \in L^{5p/(3+p)}_{t,x} \text{ (ONLY for } p \leq 2)$$

Weak (L^p) space: $W_p := \{w : w^{p/2} \in L^\infty(L^2) \cap L^2(H^1)\}$.
Strong (L^q) space:

$$X_q := \{w \in C([0,T]; W^{2-2/q,q}) \cap L^q(W^{2,q}) : \partial_t w \in L^q(L^q)\}.$$

Main results

- Global in time weak solution $(u, v) \in W_p \times X_2$ and convergence to constant states $(\fint u_0, \fint (u_0)^p)$ as $t \to +\infty$
- Unique global in time strong solution (1D and 2D case).

$p = 1$: [Cieslak et al, 2008]
$p = 2$: [Guillén-González, RB, Rueda-Gomez. CAMWA 2020]
$1 < p < 2$: [Guillén-González, RB, Rueda-Gomez. Submitted]

Open problem: In 3D, Blow-up or Global regularity ?

Optimal control problem

$$\min J(u,v,f) = \frac{\gamma_u}{r}\int_0^T\!\!\int_\Omega |u-u_d|^r + \frac{\gamma_v}{2}\int_0^T\!\!\int_\Omega |v-v_d|^2 + \frac{\gamma_u}{q}\int_0^T\!\!\int_{\Omega_c} |f|^q$$

with $r = \begin{cases} 2 & \text{in 2D,} \\ 20/7 & \text{in 3D,} \end{cases}$ and $q = \begin{cases} 2+\varepsilon & \text{in 2D,} \\ 4 & \text{in 3D,} \end{cases}$

subject to (control) $f \in \mathcal{F}$ a closed convex of $L^q((0,T)\times\Omega_c)$ and (state) (u,v) the strong solution of

$$\begin{cases} \partial_t u - \Delta u - \nabla\cdot(u\nabla v) = 0 \\ \partial_t v - \Delta v = u^p + f\, v\, 1_{\Omega_c} \end{cases}$$

Results:

1. Existence of global optimal solution
2. First order optimality system.

References:

$p = 1$ and 2D: [Guillén-González, Mallea-Zepeda, RB. COCV 20]

$p = 1$ and 3D: [Guillén-González, Mallea-Zepeda, RB. SICON 20]

2D: Well-posedness of State problem

$\forall f \in L^{2+\varepsilon}(Q_c)$, $\exists!(u,v) \in X_2 \times X_{2+\varepsilon}$ strong solution. Moreover,

$$\|(u,v)\|_{X_2 \times X_{2+\varepsilon}} \leq C(\|f\|_{L^{2+\varepsilon}(Q_c)})$$

Keys of the proof:

1. Leray-Schauder Theorem: $R : (\overline{u}, \overline{v}) \to (u, v)$ solving
   1. v : $\partial_t v - \Delta v = \overline{u}_+ + f\,\overline{v}_+\,1_{\Omega_c}$,
   2. u : $\partial_t u - \Delta u = \nabla \cdot (\overline{u}_+ \nabla v)$
2. Energy estimates of (possible) fixed-points

$$(u,v) = \lambda R(u,v), \quad \lambda \in [0,1]$$

3. Bootstrapping argument, via L^p regularity of the Heat-Neumann problem

3D: Existence of State problem. Regularity criterium (uniqueness)

$\forall f \in L^4(Q_c)$, $\exists (u, v) \in W_2 \times X_2$ a weak solution.
If $u \in L^{20/7}(Q)$ then $(u, v) \in X_2 \times X_4$ is the unique strong solution

$$\|(u, v)\|_{X_2 \times X_4} \le C(\|f\|_{L^4(Q_c)})$$

1. Regularization. $\forall \varepsilon > 0$, let $(u^\varepsilon, z^\varepsilon)$ the solution of

$$\begin{cases} \partial_t u - \Delta u - \nabla \cdot (u \nabla v^\varepsilon(z)) = 0 \\ \partial_t z - \Delta z = u + f\, v^\varepsilon(z)_+ 1_{\Omega_c} \end{cases}$$

   with $v^\varepsilon(z)$ the solution of: $v - \varepsilon \Delta v = z$, + BCs
2. Existence $(u^\varepsilon, z^\varepsilon)$ via Leray-Schauder Theorem.
3. ε-independent energy estimates
4. Passing to the limit as $\varepsilon \to 0$ $(z^\varepsilon - v(z^\varepsilon) \to 0)$, one has existence of weak solution.
5. Regularity criterium $u \in L^{20/7}(Q)$. Bootstrapping argument.

Existence of Global minimum

Admisible set

$$\mathcal{U}_{ad} = \{(u, v, f) \in X_2 \times X_q \times \mathcal{F} : \ (u, v, f) \text{ strong solution}\}$$

Hypothesis:

- In 3D, $\mathcal{U}_{ad}$ in not empty ($u \in L^{20/7}(Q)$).
  OK if the control acts on the whole domain ($\Omega_c = \Omega$) and $v_0 \geq v_{0,min} > 0$ in Ω.
- $\gamma_u > 0$ and, either $\gamma_f > 0$ or $\gamma_f = 0$ and $\mathcal{F}$ bounded in $L^q(Q_c)$,

Then, there exists global optimal solution (minimizing sequences).

Step 1: Linearized problem

Any $(\widehat{u}, \widehat{v}, \widehat{f}) \in X_2 \times X_q \times L^q(Q_c)$ is a **"regular point"**,
i.e. for any data $(g_1, g_2) \in L^2 \times L^q$, the linearized problem around $(\widehat{u}, \widehat{v}, \widehat{f})$:

$$\left\{ \begin{array}{l} \partial_t U - \Delta U - \nabla \cdot (U \nabla \widehat{v} + \widehat{u} \nabla V) = g_1 \\ \partial_t V - \Delta V - U - \widehat{f}\, V\, 1_{\Omega_c} - F\, \widehat{v}\, 1_{\Omega_c} = g_2, \\ U(0) = 0 = V(0), \quad + BCs \end{array} \right.$$

has a solution $(U, V, F) \in X_2 \times X_q \times \mathcal{C}(\mathcal{F})$
(with $\mathcal{C}(\mathcal{F}) = \{\theta(f - \widehat{f}) : \ f \in \mathcal{F}, \ \theta \geq 0\}$).

Step 2: Lagrange Multipliers

Existence of Lagrange multipliers $(\widehat{\lambda}, \widehat{\eta}) \in L^2(Q) \times L^{q'}(Q)$, which is a (very-weak) solution of the variational problem:

$$\iint \widehat{\lambda}\Big(\partial_t U - \Delta U - \nabla \cdot (U \nabla \widehat{v})\Big) - \iint \widehat{\eta} U = \iint \widehat{J}_u U,$$

$$\iint \widehat{\lambda}\Big(-\nabla \cdot (\widehat{u} \nabla V)\Big) + \iint \widehat{\eta}\Big(\partial_t V - \Delta V - \widehat{f}\, V\, 1_{\Omega_c}\Big) = \iint \widehat{J}_v V,$$

$$\iint_{\Omega_c} \widehat{\eta}\Big(-\widehat{v}\,(F - \widehat{f})\Big) \geq \iint_{\Omega_c} \widehat{J}_f (F - \widehat{f}),$$

$\forall\, (U, V, F) \in X_2 \times X_q \times \mathcal{F}$ with $U(0) = 0 = V(0)$

Step 3: Regularity of Lagrange multipliers

- Existence of strong solution of the (linear and backward) adjoint problem $(\lambda, \eta) \in X_2 \times X_{q'}$

$$\begin{cases} -\partial_t \lambda - \Delta \lambda + \nabla \widehat{v} \cdot \nabla \lambda - \eta = \widehat{J}_u \\ -\partial_t \eta - \Delta \eta - \widehat{f}\, \eta \, 1_{\Omega_c} - \nabla \cdot (\widehat{u} \nabla \lambda) = \widehat{J}_v, \\ \lambda(T) = 0 = \eta(T), \quad +BCs \end{cases}$$

- Uniqueness of a very-weak $(\widehat{\lambda}, \widehat{\eta})$ and a strong solution (λ, η) of the adjoint problem.

THEN, the Lagrange multiplier $(\widehat{\lambda}, \widehat{\eta})$ is unique and has strong regularity.

Optimality system

Coupled forward/backward optimality system: (u, v, f, λ, η) s.t.

$$\begin{cases} \partial_t u - \Delta u - \nabla \cdot (u \nabla v) = 0 \\ \partial_t v - \Delta v - u - f\, v\, 1_{\Omega_c} = 0, \\ -\partial_t \lambda - \Delta \lambda + \nabla v \cdot \nabla \lambda - \eta = J_u \\ -\partial_t \eta - \Delta \eta - f\, \eta\, 1_{\Omega_c} - \nabla \cdot (u \nabla \lambda) = J_v, \\ u(0) = u_0, \ v(0) = v_0, \ \lambda(T) = 0 = \eta(T), \quad + BCs \end{cases}$$

$$- \iint_{\Omega_c} (\eta v + J_f)\,(F - f) \geq 0, \quad \forall\, F \in \mathcal{F}$$

Conclusions

1. Bilinear control introduces a non-regular term in the chemotaxis PDE problem. In particular, classical solutions (Amann's argument) cannot be used.
   Regularization + compactness $\Rightarrow$ $\exists$ weak solutions.
2. A regularity criterium must be deduced implying strong regularity for 3D domains (and continuous dependence).
   In 1D or 2D domains, any weak solution is the unique strong solution.
3. Existence of global optimal solution, via minimizing sequence
4. Existence of (very weak) Lagrange Multipliers
5. Regularity and uniqueness of Lagrange Multiplier;
   very-weak vs strong uniqueness.

Work in progress

1. Chemoattraction + production vs logistic reaction for u:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla v) = u(1-u) \\ \partial_t v - \Delta v = u + f\, v\, 1_{\Omega_c} \end{cases}$$

2. Chemoattraction vs Consumption:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0 \\ \partial_t v - \Delta v = -u\, v + f\, v\, 1_{\Omega_c} \end{cases}$$

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