Minimum supports of eigenfunctions of graphs

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Let $G = (V, E)$ be a graph and let $\lambda$ be an eigenvalue of $G$. The set of neighbors of a vertex $x$ is denoted by $N(x)$. A function $f : V \rightarrow \mathbb{R}$ is called a $\lambda$-eigenfunction of $G$ if $f \not\equiv 0$ and the equality

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds for any vertex $x \in V$. The support of a function $f : V \rightarrow \mathbb{R}$ is the set $S(f) = \{x \in V \mid f(x) \neq 0\}$. A $\lambda$-eigenfunction of $G$ is called optimal if it has the minimum cardinality of the support among all $\lambda$-eigenfunctions of $G$. 
Main problem

MS-problem

Let $G$ be a graph and let $\lambda$ be an eigenvalue of $G$. Find the minimum cardinality of the support of a $\lambda$-eigenfunction of $G$.

MS-problem is closely related to the intersection problem of two combinatorial objects and to the problem of finding the minimum cardinality of combinatorial trades [1,2].

Results on MS-problem

MS-problem has been studied for the following families of graphs:

- bilinear forms graphs (Sotnikova, 2019)
- cubical distance-regular graphs (Sotnikova, 2018)
- Doob graphs (Bespalkov, 2018)
- Grassmann graphs (Cho 1999; Krotov, Mogilnykh, Potapov, 2016)
- Hamming graphs (Vorob’ev, Krotov, 2014; Krotov 2016; Valyuzhenich, Vorobev, 2019; Valyuzhenich 2021)
- Johnson graphs (Vorob’ev, Mogilnykh, Valyuzhenich, 2018)
- Paley graphs (Goryainov, Kabanov, Shalaginov, Valyuzhenich, 2018)
- Star graphs (Goryainov, Kabanov, Konstantinova, Shalaginov, Valyuzhenich, 2020)
Let $\Sigma_q = \{0, 1, \ldots, q-1\}$. The Hamming graph $H(n, q)$ is defined as follows:

- the vertex set of $H(n, q)$ is $\Sigma_q^n$
- two vertices are adjacent if they differ in exactly one coordinate

The Hamming graph $H(n, q)$ has $n + 1$ distinct eigenvalues $\lambda_i(n, q) = n(q - 1) - q \cdot i$, where $0 \leq i \leq n$. 
MS-problem for the Hamming graph

- MS-problem for the Hamming graph $H(n, 2)$ is solved for all eigenvalues in [3].
- MS-problem for the Hamming graph $H(n, q)$ is solved for all eigenvalues and $q \geq 3$ in [4,5].
- Moreover, in [4] a characterization of optimal $\lambda_i(n, q)$-eigenfunctions of $H(n, q)$ was obtained for $q \geq 3$, $i \leq \frac{n}{2}$ and $q \geq 5$, $i > \frac{n}{2}$.

Optimal eigenfunctions of the Hamming graph

So, the problem of a characterization of optimal $\lambda_i(n, q)$-eigenfunctions of $H(n, q)$ is open for the following cases:

- $q = 2$
- $q = 3$ and $i > \frac{n}{2}$
- $q = 4$ and $i > \frac{n}{2}$
In 2016 D. Krotov [3] proved that the minimum cardinality of the support of a $\lambda_i(n, 2)$-eigenfunction of $H(n, 2)$ is $\max(2^i, 2^{n-i})$. He also gave examples of optimal $\lambda_i(n, 2)$-eigenfunctions of $H(n, 2)$. These optimal eigenfunctions can be constructed as a tensor product of several optimal eigenfunctions defined on the vertices of the Hamming graphs of diameter not greater than two.

Suppose $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs. Let $f_1 : V_1 \to \mathbb{R}$ and $f_2 : V_2 \to \mathbb{R}$. Denote $G = G_1 \square G_2$. We define the tensor product $f_1 \cdot f_2$ on the vertices of $G$ by the following rule:

$$(f_1 \cdot f_2)(x, y) = f_1(x)f_2(y)$$

for $(x, y) \in V(G) = V_1 \times V_2$. 
Optimal eigenfunctions in $H(2, 2)$ and $H(1, 2)$

The set $A$ consists of two optimal 0-eigenfunctions of $H(2, 2)$:

The sets $B$ and $C$:
Optimal eigenfunctions in $H(n, 2)$

Optimal $\lambda_i(n, 2)$-eigenfunctions of $H(n, 2)$ for $i \leq \frac{n}{2}$:

Optimal $\lambda_i(n, 2)$-eigenfunctions of $H(n, 2)$ for $i > \frac{n}{2}$:
Main result

Theorem (V., 2021)

If $i \leq \frac{n}{2}$, then any optimal $\lambda_i(n, 2)$-eigenfunction of $H(n, 2)$ is the tensor product of $i$ functions from the set $A$ and $n - 2i$ functions from the set $B$ up to a permutation of coordinate positions and the multiplication by a scalar.

If $i > \frac{n}{2}$, then any optimal $\lambda_i(n, 2)$-eigenfunction of $H(n, 2)$ is the tensor product of $n - i$ functions from the set $A$ and $2i - n$ functions from the set $C$ up to a permutation of coordinate positions and the multiplication by a scalar.
Thank you for your attention!