Exact spectral asymptotics of fractional processes and its applications

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Outline

1. Introduction
   - Problem statement
   - Why it is interesting
   - What is well known
   - Essence of our work

2. Main Results
   - Spectral Asymptotics
   - Some Applications

3. Description of the method

4. Concluding Remarks

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Objects of study

We consider

- Zero mean Gaussian process $X = (X_t; t \in [0, 1])$ with covariance function
  \[ K(s, t) = \mathbb{E}X_sX_t, \quad s, t \in [0, 1] \]

- Covariance operator
  \[ f \mapsto (Kf)(t) := \int_0^1 K(s, t)f(s)ds, \quad t \in [0, 1] \]
A spectral problem

Problem

Given a covariance operator $K$, compute its eigenvalues and eigenfunctions, i.e. solve the equation:

$$(K\varphi_n)(t) = \lambda_n \varphi_n(t), \quad t \in [0, 1].$$

Unfortunately $\lambda_n$ and $\varphi_n$ are rarely available in closed form.
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Applications

- Karhunen-Loeve expansion
- Equivalence and orthogonality of Gaussian measures
- Approximate sampling from heavy tailed distributions
- Numerical solutions of stochastic equations
- Exact asymptotics of the small ball probabilities
- Asymptotics of solutions of integral equations
- ...
Some comments: numerical methods

Numerical approximations...

- ...give reliable results for several first eigenvalues
- ...do not work for large values of $n$

Numerical relative error for eigenvalues of Brownian motion
Small $L_2$-ball probabilities

**Problem**

Given a process $X = (X_t, t \in [0, 1])$, find the asymptotics of

$$P(\|X\| \leq \varepsilon) \quad \text{as } \varepsilon \to 0.$$ 

To solve it, we need asymptotics

$$\lambda_n = \sum_{j=1}^{k} c_j n^{-d_j} + O(n^{-\gamma}),$$

where $\gamma - d_1 > 1$, $0 \leq d_j - d_1 \leq 1$. 

Why it is non trivial

Discontinuity of the second order term

Let $K(s, t) = s \wedge t - \varepsilon s t \quad \varepsilon \in [0, 1]$. Then

$$\lambda_n = \nu_n^{-2}$$

where

- $\nu_n = n\pi - \frac{1}{2}\pi + O(n^{-1})$ when $\varepsilon \in [0, 1)$
- $\nu_n = n\pi$ when $\varepsilon = 1$. 

Solutions of second kind integral equations

Two variants of the problem

- \((\varepsilon \to 0)\) Singularity perturbed integral equations
  \[
  \varepsilon u_\varepsilon(x) + (Ku_\varepsilon)(x) = f(x), \quad x \in [0, 1]
  \]

- \((T \to \infty)\) Large time behaviour of the solution
  \[
  u_T(x) + (Ku_T)(x) = f(x), \quad x \in [0, T]
  \]

They arise in

- optimal linear filtering/interpolation problems
- statistical inference of processes
Likelihood type estimates for mixed fBm noise systems

Singular perturbations

Fix $\varepsilon > 0$ and let $g_\varepsilon$ be the solution of the equation:

$$\varepsilon g_\varepsilon(u) + \frac{d}{du} \int_0^1 g_\varepsilon(v) |u-v|^{2H-1} \text{sign}(u-v) dv = 1, \; u \in [0,1],$$

An important question

What can we say about $g_\varepsilon$ when $\varepsilon \to 0$?
Likelihood type estimates for mixed fBm noise systems

Singular perturbations

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Spectral asymptotics

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State-of-the-art

Asymptotics for Brownian motion and related processes

- For the Brownian motion $K(s, t) = \min(s, t)$
  \[
  \lambda_n = \frac{1}{(n - \frac{1}{2})^2 \pi^2} \quad \text{and} \quad \varphi_n(t) = \sqrt{2} \sin \left( n - \frac{1}{2} \right) \pi t
  \]

- Similar results for related processes (Brownian bridge, Ornstein–Uhlenbeck process, etc.)

State-of-the-art techniques

- Reduce the original eigenproblem to a **classical** Sturm-Liouville problem.
- This does not work for a "long memory" processes.
State-of-the-art

Asymptotics for fractional Brownian motion

- For the fractional Brownian motion the leading asymptotic term

\[ \lambda_n = \frac{\sin(\pi H) \Gamma(2H + 1)}{(n\pi)^{2H+1}} + o \left( n^{-\left(\frac{2H+2}{4H+5}\cdot\frac{4H+3}{4H+5}\right)} + \delta \right) \]


- Nothing was known about the eigenfunctions.
## Classical and Fractional Sturm-Liouville problem

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Brownian motion $W_t$

**Spectral problem:**

$$\int_0^1 x \wedge y \phi(y) \, ds = \lambda \phi(x), \quad x \in [0, 1].$$

**Sturm-Liouville problem (classical):**

$$-\phi'' = \lambda^{-1} \phi, \quad x \in [0, 1]$$

$$\phi(0) = \phi'(1) = 0$$

**Solution to spectral problem:**

$$\lambda_n = \frac{1}{(n - \frac{1}{2})^2 \pi^2} \quad \text{and} \quad \phi_n(t) = \sqrt{2} \sin \left( (n - \frac{1}{2}) \pi t \right)$$
Details for Riemann-Liouville process

\[
\int_0^t (t - s)^{\alpha - 1} dW_s
\]

**Sturm-Liouville problem (fractional):**

\[
cD_1^\alpha - cD_0^\alpha u(x) = \lambda^{-1} u(x), \quad x \in [0, 1],
\]

\[
u(0) = 0, \quad cD_0^\alpha u(1) = 0,
\]

with the left and right Caputo derivatives of order \( \alpha \in (0, 1) \).

**Spectral problem:**

\[
\int_0^1 K(x, y)f(y)dy = \lambda f(x), \quad x \in [0, 1],
\]

with the kernel

\[
K(x, y) = \frac{1}{\Gamma(\alpha)^2} \int_0^{x \wedge y} (x - y)^{\alpha-1}(y - t)^{\alpha-1} dt,
\]
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## Essence of our work

### Problem solved

- Obtain asymptotics of
  - eigenfunctions with respect to the uniform norm
  - eigenvalues up to the second/third... order terms
for a large class of fractional processes.

### Main tool

- Spectral problem reduced to an equivalent
  integro-algebraic system of equations, more amenable
  to asymptotic analysis
- The method is a transposition of techniques for
  Riemann boundary value problems

Essence of our work

Problem solved

Obtain asymptotics of

- eigenfunctions with respect to the uniform norm
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Main tool

- Spectral problem reduced to an equivalent integro-algebraic system of equations, more amenable to asymptotic analysis
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Class C of operators

The method, in principle, applies to
- operators with difference kernels representable as
  \[ K(u) = \int_0^\infty \kappa(t)e^{-t|u|}dt, \]
  i.e. having inverse Laplace transform.
- their compositions with the integration operator
- linear combinations (mixed Gaussian processes)

But the implementation of the method
- is very specific to the fine structure of the kernel
- requires different tricks and leads to entirely unexpected outcomes
Operators of class $C, 1$

**fBm "derivative"**

$$(Kf)(t) = \frac{d}{dt} \int_0^1 f(y) H|t - y|^{2H-1} \text{sign}(t - y) dy.$$ 

**fBm**

$$(Kf)(t) = \int_0^1 K(t, y)f(y) dy,$$

where

$$K(x, y) = \frac{1}{2} \left( x^{2H} + y^{2H} - |x - y|^{2H} \right), \quad x, y \in [0, 1].$$
Operators of class $C_1$.

**fBm "derivative"**

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where

$$K(x, y) = \frac{1}{2} \left(x^{2H} + y^{2H} - |x - y|^{2H}\right), \quad x, y \in [0, 1].$$
Operators of class $C$, II

**fBm integral**

$$K(x, y) = \int_0^t \int_0^s \frac{1}{2} \left( u^{2H} + v^{2H} - |v - u|^{2H} \right) dudv$$

**Fractional Ornstein–Uhlenbeck**

$$K(x, y) = \int_0^t e^{\beta(t-v)} \frac{d}{dv} \int_0^s H|v-u|^{2H-1} \text{sign}(v-u) e^{\beta(s-u)} dudv.$$
Operators of class $C$, II

**fBm integral**

\[
K(x, y) = \int_0^t \int_0^S \frac{1}{2} \left( u^{2H} + v^{2H} - |v - u|^{2H} \right) dudv
\]

**Fractional Ornstein–Uhlenbeck**

\[
K(x, y) = \int_0^t e^{\beta(t-v)} \frac{d}{dv} \int_0^S H|v-u|^{2H-1} \text{sign}(v-u) e^{\beta(s-u)} dudv.
\]
Operators of class $C$, III

Mixed fBm

Mixture with an independent standard Brownian motion $B$

$$X_t = B_t + B_t^H, \quad t \in [0, 1],$$

$$K(x, y) = s \wedge t + c_\alpha \int_0^t \int_0^s |u - v|^{-\alpha} dudv.$$

with $\alpha := 2 - 2H \in (0, 1)$ and $c_\alpha := (1 - \frac{\alpha}{2})(1 - \alpha)$
"Derivative" of the fBm

**Eigenvalues**

\[ \lambda_n = \sin(\pi H) \Gamma(2H + 1) \nu_n^{1-2H}, \quad n = 1, 2, \ldots, \text{where} \]

\[ \nu_n = \left( n - \frac{1}{2} \right) \pi + \frac{1 - 2H}{4} \pi + O(n^{-1}) \quad \text{as} \ n \to \infty. \]

**Eigenfunctions**

\[ \varphi_n(x) = \sqrt{2} \cos \left( 2\nu_n x - \frac{1 + \alpha}{8} \pi \right) + \]

\[ \sqrt{\left| \alpha - 1 \right|} \frac{1}{\pi} \int_0^\infty |\rho_0(\tau)| \left( e^{-2\nu_n x \tau} - (-1)^n e^{-2\nu_n (1-x) \tau} \right) d\tau \]

\[ + n^{-1} r_n(x), \quad x \in [0, 1], \text{ with } \alpha = 2 - 2H \in (0, 2) \setminus \{1\} \]
fBm and fOU: Eigenvalues Asymptotics

Eigenvalues

The eigenvalues satisfy

\[ \lambda_n = \sin(\pi H) \Gamma(2H + 1) \frac{\nu_n^{1-2H}}{\nu_n^2 + \beta^2}, \quad n = 1, 2, \ldots \]

where \( \nu_n \) is a sequence with the following asymptotics

\[ \nu_n = (n - \frac{1}{2})\pi - \frac{(H - 1/2)^2}{(H + 1/2)} \frac{\pi}{2} + O_\beta(n^{-1}) \quad \text{as } n \to \infty. \]
fBm and fOU: Eigenfunctions Asymptotics

Eigenfunctions

The corresponding normalized eigenfunctions admit the approximation

$$\varphi_n(x) = \sqrt{2} \sin \left( \nu_n x + \frac{H - 3/2}{4} \frac{H - 1/2}{H + 1/2} \pi \right) +$$

$$\int_0^\infty f_0(u) e^{-x \nu_n u} du + (-1)^n \int_0^\infty f_1(u) e^{-(1-x) \nu_n u} du$$

$$+ \nu_n^{-1} r_n(x),$$

where the residual $r_n(x)$ is bounded by a constant, depending only on $H$, and $f_j(u)$ is an explicit function.
A typical shape of the eigenfunctions

Figure: $\varphi_{10}$ and $\varphi_{11}$ for $H = \frac{1}{4}$ (left) and $H = \frac{3}{4}$ (right)
Integrated fBm Eigenvalues Asymptotics

**Eigenvalues**

The eigenvalues of covariance operator of integrated fBm satisfy

\[ \lambda_n = \sin(\pi H) \Gamma(2H + 1) \nu_n^{-2H-3} \quad n = 1, 2, \ldots \]

where

\[ \nu_n = \left(n - \frac{1}{2}\right)\pi - \frac{(H - 1/2)(H + 1/2)}{H + 3/2} \frac{\pi}{2} + O(n^{-1}). \]
Integrated fBm Eigenfunctions Asymptotics

The corresponding eigenfunctions admit the approximation

\[ \varphi_n(x) = \varphi_n^{(1)}(x) + \varphi_n^{(2)}(x) + \varphi_n^{(3)}(x) + n^{-1} r_n(x) \]

where \( r_n(x) \) is bounded uniformly in both \( n \in \mathbb{N} \) and \( x \in [0, 1] \) and

\[ \varphi_n^{(1)}(x) = \sqrt{2} \cos \left( \nu_n x + \frac{2H + 1}{8} \pi - \frac{H - 1/2}{H + 3/2} \pi \right) \]

\[ \varphi_n^{(2)}(x) = \int_0^\infty \rho_0(t) \left( Q_0(t)e^{-t\nu_n x} - (-1)^n Q_1(t)e^{-t\nu_n(1-x)} \right) dt \]

\[ \varphi_n^{(3)}(x) = C_6 e^{-c\nu_n x} \cos \left( s\nu_n x + \kappa_0 \right) + C_7 e^{-c\nu_n(1-x)} \cos \left( s\nu_n(1 - x) + \kappa_1 \right) \]
Mixed fBm spectral problem

**Eigenvalues:**

$$
\lambda_n = \frac{1}{\nu_n^2} + \frac{\sin(\pi H)\Gamma(2H + 1)}{\nu_n^{2H+1}}, \quad n = 1, 2, \ldots
$$

where

$$
\nu_n = \nu_n^{BM} 1_{\{H > \frac{1}{2}\}} + \nu_n^{fBM} 1_{\{H \leq \frac{1}{2}\}} + O(n^{-|2H-1|})
$$

**Eigenfunctions:**

$$
\varphi_n(x) = \varphi_n^{BM}(x) 1_{\{H > \frac{1}{2}\}} + \varphi_n^{fBM}(x) 1_{\{H \leq \frac{1}{2}\}} + \frac{1}{n^{2H-1}} r_n(x)
$$
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Integro-differential Equation

\[ \varepsilon g_\varepsilon(u) + \frac{d}{du} \int_0^1 g_\varepsilon(v)|u-v|^{2H-1}\text{sign}(u-v)dv = 1, \quad u \in [0,1], \]

Corollary from our asymptotics

Convergence results for solutions:

- weak convergence with rate \( \varepsilon \)
- \( L^2 \) convergence with a rate depending on \( H \)
- boundary layer construction with \( \frac{1}{\sqrt{\varepsilon}} \) rate
Exact Small Balls Probability, fBm

For all $H \in (0, 1)$,

$$
\mathbb{P}(\|B^H\|_2 \leq \varepsilon) \simeq \varepsilon^{\gamma(H)} \exp \left(-\beta(H)\varepsilon^{-\frac{1}{H}}\right), \quad \varepsilon \to 0
$$

where

$$
\beta(H) = \frac{H}{(2H + 1)^{\frac{2H+1}{2H}}} \left(\frac{\sin(\pi H)\Gamma(2H + 1)}{\left(\sin \left(\frac{\pi}{2H+1}\right)\right)^{2H+1}}\right)^\frac{1}{2H}
$$

and

$$
\gamma(H) = \frac{1}{2H} \left(\frac{3}{4} + H^2 - H + \frac{1}{2}\right),
$$
Exact Small Bals Probability, mixed fBm

\[ \mathbb{P}(\|\widetilde{B}\|_2 \leq \varepsilon) \sim \varepsilon^\gamma \exp \left( - \sum_{k=0}^{\lfloor 2H-1 \rfloor} \beta_k \varepsilon^{H^\frac{1}{2}} (k|2H-1|-1) \right) \]

where \( \gamma \) and \( \beta_k, k = 0, 1, 2, \ldots \) are explicitly defined functions of \( H \).
Linear filtering of fBm

Large scale asymptotics, $T \to \infty$

$$Y_t = \mu \int_0^t B_s^H ds + B_t, \quad t \in [0, T]$$

The steady state filtering error is given by

$$\lim_{T \to \infty} P_T = \left( \frac{\sin(\pi H) \Gamma(2H + 1)}{\sin \frac{\pi}{2H+1}} \right)^{\frac{1}{2H+1}} \mu^{-\frac{4H}{2H+1}}$$
Linear filtering of fOU

Small noise asymptotics, $\varepsilon \to 0$

$$Y_t = \mu \int_0^t X_s ds + \sqrt{\varepsilon} B_t, \quad t \in [0, T]$$

The high signal-to-noise filtering error is given by

$$P_{\varepsilon}(t) \approx \left(\frac{\varepsilon}{\mu^2}\right)^{\frac{2H}{1+2H}} \frac{\sin(\pi H)\Gamma(2H + 1)}{\sin \frac{\pi}{2H+1}} \begin{cases} \frac{1}{2H+1} & t < T \\ 1 & t = T \end{cases}$$
Approach in a nutshell, I

For the Laplace transform (a priori analytic function!)

\[ \hat{\varphi}(z) := \int_0^1 \varphi(x)e^{-zx} \, dx, \quad z \in \mathbb{C} \]

find an expression with \textit{handy} singularities.
Reduce the original spectral problem to the solution of **Riemann-Hilbert boundary value problem** of finding two analytical on the cut plane $\mathbb{C} \setminus \mathbb{R}_{>0}$ functions $\Phi_0(z)$ and $\Phi_1(z)$ that satisfy

- **boundary condition** on $\mathbb{R}_{>0}$
- **a priori estimates** at $z = 0$ and **polynomial growth rate** at $z \to \infty$
- certain **algebraic conditions** on the imaginary axis.
Sketch of proof

- Using the particular structure of the eigenproblem obtain

\[ \tilde{\varphi}(z) = P(z) - \frac{Q(z)}{\Lambda(z)} \left( e^{-z} \Phi_1(-z) + \Phi_0(z) \right), \quad z \in \mathbb{C} \]

where

- \( \Phi_0 \) and \( \Phi_1 \) are analytic on \( \mathbb{C} \setminus \mathbb{R}_+ \)
- \( \Lambda(z) \) is an explicit function
- \( P(z) \) and \( Q(z) \) are polynomials of a finite degree

- A calculation reveals that the function \( \Lambda(z) \) has
  - has a finite number of zeros \( z_1(\lambda), \ldots, z_k(\lambda) \)
  - jump discontinuity along real line
Sketch of proof

- Removal of the discontinuity gives conditions on the limit values

\[ \Phi_0^\pm(t) = \lim_{z \to t^\pm} \Phi_0^\pm(z) \quad \text{and} \quad \Phi_1^\pm(t) = \lim_{z \to t^\pm} \Phi_1^\pm(z), \quad t \in \mathbb{R}_+ \]

in the form of a coupled pair of nonhomogeneous Hilbert BVPs

- The Hilbert BVPs decouple and their solutions lead to integral equations for \( \Phi_0^\pm(t) \) and \( \Phi_1^\pm(t) \), \( t \in \mathbb{R}_+ \)

- Solutions of these equations determine \( \Phi_0(z) \) and \( \Phi_1(z) \) on the whole cut plane \( \mathbb{C} \setminus \mathbb{R}_+ \) (and in turn the Laplace transform \( \hat{\varphi}(z) \))

- Removal of the poles gives algebraic constraints on \( \Phi_0 \) and \( \Phi_1 \)
Sketch of proof

Reduction

At this stage the original eigenproblem reduces to a system of coupled integral and algebraic equations!
Sketch of proof

- The integro-algebraic system has countably many solutions, whose structure is revealed asymptotically as the algebraic part of the solution tends to $+\infty$

- The eigenvalue asymptotics is extracted from the algebraic part of the solutions

- The eigenfunctions asymptotics is obtained by Laplace transform inversion of the integral part of the solutions
Work in progress

- Spectrum of weakly singular operators, in particular with logarithmic singularities
- Spectrum of fBm type processes (fractional Brownian sheet, multidimensional case...) and applications
- Estimation of $H$ in the mixed fBm processes and the standard filtering setting
Some references


