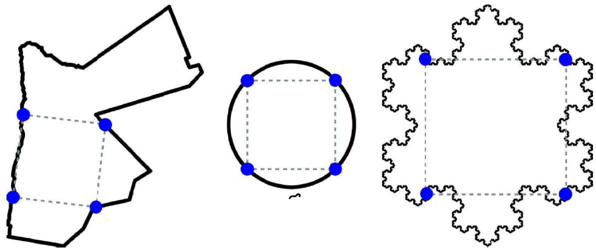
# 8ECM, MS11: Inscribed rectangles & non-orientable slice surfaces

(based on arXiv:2003.01590, with M. Golla)

Square peg problem (Töplitz, 1911): Does every Jordan curve $\Gamma$ have an inscribed square?



Answer: • In general open.

• Schnirelman 1929: Yes, if $\Gamma$ is smooth.

New proof by Hugelmeyer using the following:

Fact: If $F \subseteq S^1 \times B^3$ is a smooth surface with $\partial F = T(4,4k+1) \subseteq S^1 \times S^2$, then $b_1(F) \geq 2$.

Dream: ~~smooth~~ loc. flat & make proof work

(In the news: Greene–Lobb: Every smooth $\Gamma$ has inscribed rectangles of all aspect ratios.)

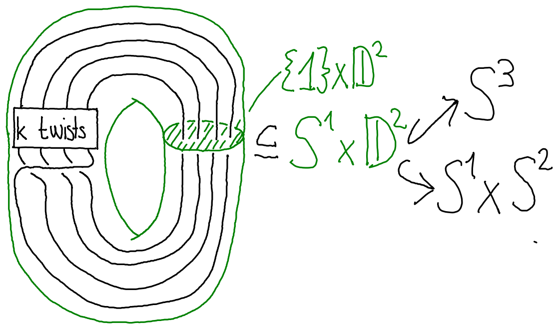
3.5D Topology: Fix $b \in \mathbb{N}_0$ & $X$ a 4-dimensional smooth manifold with boundary $\partial X \neq \emptyset$.

Q: For which knot $K \subseteq \partial X$, $\exists F \subseteq X$ a surface (loc. flat/smooth) with $\partial F = K$ & $b_1(F) \leq b$.

Thm. (F.-Golla): For $n \in \mathbb{N}$ not a square: If $F \subseteq S^1 \times B^3$ is a loc. flat surface with $\partial F = T(2n, 2nk \pm 1) \subseteq S^1 \times S^2$, then $b_1(F) \geq 2$.

$T(p,q)$:

$T(4,4k+1) =$




## Why is Q interesting for dim(X)=4? (b=0)

Dim(X) ≠ 4:

Dehn's Lemma: For $K \subseteq \partial X$, there $\exists F \subseteq X$ ($F \cong D^2$) smooth with $\partial F = K$ iff $K$ is nullhomotopic.

$X = B^4$: Q $\Leftrightarrow$ Which knots $K \subseteq S^3$ are slice in $B^4$?

Dim(X)=4: Topological (= loc. flat) vs. smooth $F \subseteq X$:

Topology: A version of Dehn's Lemma holds: Freedman's Disk Embedding Theorem (DET).

Smooth: More obstructions (e.g. from Heegaard-Floer).

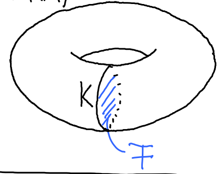
Concrete quantities to differentiate between top. and smooth (for $X = B^4$):

Let $K \subseteq S^3$ be a knot.

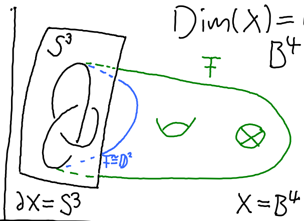
$\gamma_4(K) := \min\{b_1(F) \mid F \subseteq B^4, \partial F = K \subseteq S^3\}$ (smooth surface)

$\gamma_4^{top}(K) := \min\{b_1(F) \mid F \subseteq B^4, \partial F = K \subseteq S^3\}$ (loc. flat surface)

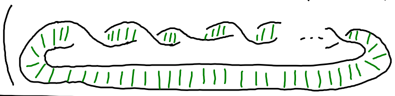
Dim(X)=3, $\partial X = S^1 \times S^1$, $X = S^1 \times D^2$




Dim(X)=4, $B^4$




Example: $\gamma_4^{top}(T(2,2n+1)) = \gamma_4(T(2,2n+1)) = 1$ for $n \geq 1$.




Open problem: $\gamma_4^{top}(K) \leq 10$ $\forall$ knots $K$?

Batson (2012): $\gamma_4(T(2n, 2n-1)) = n-1$ for $n \in \mathbb{N}$.

Tool in the proof: Heegaard-Floer.

Cor. (of Batson (2012)): For $n \in \mathbb{N}$. If $F \subseteq S^1 \times B^3$ is a smooth surface with $\partial F = T(2n, 2n-1) \subseteq S^1 \times S^2$, then $b_1(F) \geq n-1$.

Prop. (F.-Golla): For $n \geq 5$.

$\gamma_4^{top}(T(2n, 2n-1)) \leq n-2 < n-1$.

Tool in the proof: DET.

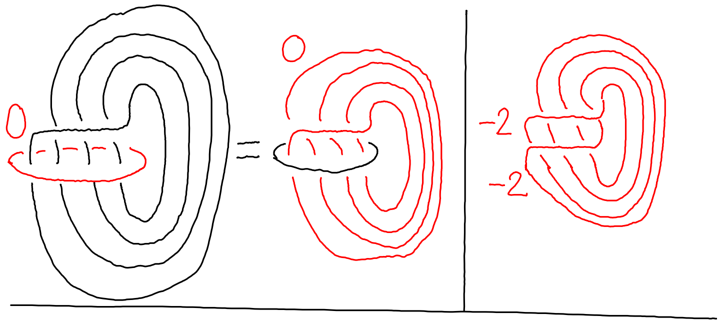
**Thm. (F.-Golla):** For $n \in \mathbb{N}$ not a square. If $F \subseteq S^1 \times B^3$ is a loc. flat surface with $\partial F = T(2n,1)$, then $b_1(F) \neq 1$.

Idea of the proof:

- Take $F \subseteq S^1 \times B^3$ loc. flat with $\partial F = T(2n,1) \subseteq S^1 \times S^2$ & $b_1(F) = 1$.
- Set $W := \Sigma(F)$ — the double branched cover of $B^4$ along $F$. & check $\partial W = \Sigma(T(2n,1)) \overset{⊛}{\cong} S^3_{(-n,-n)}(T(2n,2))$
- $Z := W \cup C$, where $C = \mathrm{Trace}_{(-n,-n)}(T(2n,2))$ $= B^4 \cup$ 2-handle $\cup$ 2-handle
- $Z$ is a closed top. 4-manifold with
  - $b_2(Z) = 1$ (homology calculation using $b_1(F) = 1$).
  - $\exists\, x \in H_2(Z;\mathbb{Z})$ with $\langle x, x \rangle = n$ (since $\exists\, S \subseteq C$ with $S \cdot S = -n$.)
- $(H_2(Z;\mathbb{Z})/\mathrm{Tor}, \langle \cdot,\cdot \rangle) \overset{\varphi}{\cong} (\mathbb{Z}, \langle \pm 1 \rangle)$ $\implies \pm\varphi(x)\varphi(x) = n \implies \ell^2 = n$ for $\ell = |\varphi(x)| \in \mathbb{N}$ ▨

⊛ (n=2)

$T(2n,1) \subseteq S^1 \times S^2 \xrightarrow{\Sigma} \Sigma(T(2n,1))$




Thanks for listening