Blow-up phenomena in nonlocal eigenvalue problems: when theories of L^1 and L^2 meet

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Nonlocal operators and related topics (MS-ID 55)

Thanks!

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Overview

- General class of operators $\mathcal{L}$
- Large $\mathcal{L}$-harmonic functions
- Notions of weak solutions
- Regularity theory
- "Large eigenfunctions"

The Fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$

- Fourier transform $\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \hat{u}(\xi)$
- Singular integral $(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x) - u(x+y)}{|y|^{n+2s}} \, dy.$
- Spectral $(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta} u(x) - u(x) \right) \frac{dt}{t^{1+s}}$
- Dirichlet-to-Neumann map (Caffarelli–Silvestre, CPDE 2007)

$$(-\Delta)^s u(x) = - \lim_{t \to 0^+} t^{1-2s} U_t(x,t),$$

where

$$\begin{cases} \operatorname{div}\left(t^{1-2s} \nabla U(x,t)\right) = 0, & (x,t) \in \mathbb{R}^n \times (0,+\infty), \\ U(x,0) = u(x), & x \in \mathbb{R}^n. \end{cases}$$

Some Fractional Laplacians in Ω

- E.g. RFL “restricted”, SFL “spectral”, CFL “censored”, ...
- $\mathcal{L} \sim (-\Delta)^{s}_{*\mathrm{FL}}$ in $\Omega \Subset \mathbb{R}^n$
- Typically $\mathcal{L}u(x) = \int_\Omega (u(x) - u(y))\mathcal{J}(x,y)\,dy + \kappa(x)u(x)$
- (K1) $\mathcal{L}^{-1}$ has Green's kernel (for $x \neq y$ in Ω)

$$\mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right) \left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right),$$

  where $\delta(x) := \mathrm{dist}(x, \Omega^c)$
- Interior regularity: $2s \in (0,2)$
- Boundary regularity: $\gamma \in (0,1]$
- Distinctions to be made later:
  - $\gamma < 2s$? (easy regularity?)
  - $\gamma > 2s - 1$? (boundary blow-up?)

RFL: Restricted Fractional Laplacian

$$\mathcal{L}u(x) = \int_\Omega (u(x) - u(y))\mathcal{J}(x,y)\,dy + \kappa(x)u(x)$$

$$\text{(K1)} \quad \mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right)\left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)$$

- Restriction: $u|_{\Omega^c} = 0$
- $(-\Delta)^s_{\mathrm{RFL}} u(x) := \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy$

$$\Longrightarrow \mathcal{J}(x,y) = \frac{1}{|x-y|^{n+2s}}, \quad \kappa(x) = \int_{\Omega^c} \frac{dy}{|x-y|^{n+2s}} \asymp \delta(x)^{-2s}$$

- $(-\Delta)^s_{\mathrm{RFL}} x_+^s = 0$ on $\mathbb{R}_+ \Longrightarrow \gamma = s$
- $2s - 1 < \gamma < 2s$ for any $s \in (0,1)$

SFL: Spectral Fractional Laplacian

$$\mathcal{L}u(x) = \int_\Omega (u(x) - u(y))\mathcal{J}(x,y)\,dy + \kappa(x)u(x)$$

$$\text{(K1)} \quad \mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right)\left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)$$

- Spectrally: $(-\Delta)^s_{\mathrm{SFL}}\varphi_j := \mu_j^s \varphi_j$ for $\begin{cases} -\Delta\varphi_j = \mu_j\varphi_j & \text{in } \Omega \\ \varphi_j = 0 & \text{on } \partial\Omega. \end{cases}$
- In terms of $\mathcal{K} =$ (heat kernel of $-\Delta$ in Ω),

$$\mathcal{J}(x,y) = \frac{1}{|\Gamma(-s)|}\int_0^\infty \mathcal{K}(t,x,y)\frac{dt}{t^{1+s}} \asymp \frac{1}{|x-y|^{n+2s}}\left(1 \wedge \frac{\delta(x)\delta(y)}{|x-y|^2}\right)$$

$$\kappa(x) = \frac{1}{|\Gamma(-s)|}\int_0^\infty \left(1 - \int_\Omega \mathcal{K}(t,x,y)\,dy\right)\frac{dt}{t^{1+s}} \asymp \delta(x)^{-2s}.$$

- Classical Hopf lemma + boundary regularity $\implies \gamma = 1$
- $2s-1 < 2s \le \gamma$ for $s \in (0, \frac{1}{2}]$, $2s-1 < \gamma < 2s$ for $s \in (\frac{1}{2}, 1)$

CFL: Censored (Regional) Fractional Laplacian

$$\mathcal{L}u(x) = \int_\Omega (u(x) - u(y))\mathcal{J}(x,y)\,dy + \kappa(x)u(x)$$

$$\text{(K1)} \quad \mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right)\left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)$$

- Censorship: $(-\Delta)^s_{\mathrm{CFL}} u(x) = \int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy$

$$\mathcal{J}(x,y) = \frac{1}{|x-y|^{n+2s}}, \qquad \kappa(x) \equiv 0.$$

- Need boundary trace $\implies$ $s \in (\frac{1}{2}, 1)$
- $(-\Delta)^s_{\mathrm{CFL}} x^{2s-1} = 0$ on $\mathbb{R}_+$ $\implies$ $\gamma = 2s-1$
- $2s-1 = \gamma < 2s$ for any $s \in (\frac{1}{2}, 1)$

The standard eigenvalue problem

$$\begin{cases} \mathcal{L}\varphi - \lambda\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{in } \mathbb{R}^n \setminus \Omega \text{ or } \partial\Omega \end{cases}$$

- Bonforte–Figalli–Vázquez (CVPDE 2018)
- (K1) $\implies 0 \le \mathbb{G}_0(x,y) \lesssim |x-y|^{-(n-2s)}$
- $\mathcal{G}_0 = \mathcal{L}^{-1} : L^2(\Omega) \xrightarrow{\text{compact}} L^2(\Omega)$ (Riesz–Fréchet–Kolmogorov)
- Discrete standard spectrum Σ

$$0 < \lambda_1 \le \lambda_2 \le \cdots \nearrow +\infty$$

- Standard eigenfunctions $\varphi_j = \lambda_j \mathcal{G}_0(\varphi_j) \implies$

$$\varphi_1 \asymp \delta^\gamma, \qquad |\varphi_j| \lesssim \delta^\gamma$$

- Energy space

$$\mathbb{H}^2_{\mathcal{L}}(\Omega) = \left\{ v \in L^2(\Omega) : \sum_{j\ge 1} \lambda_j^2 \langle v, \varphi_j \rangle^2 < +\infty \right\}$$

Large $\mathcal{L}$-harmonic functions

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u \asymp \delta^{-b} & \text{on } \partial\Omega, \end{cases} \qquad \text{e.g.} \qquad (-\Delta)^s_{\mathrm{RFL}} (1-|x|^2)_+^{s-1} = 0.$$

- Abatangelo–Gómez-Castro–Vázquez (2019)
- Precise boundary blow-up rate

$$b := \gamma - (2s-1) = \begin{cases} 1-s, & \text{RFL } (\gamma = s), \\ 2-2s, & \text{SFL } (\gamma = 1), \\ 0, & \text{CFL or } -\Delta \ (\gamma = 2s-1). \end{cases}$$

- (K2) Martin (boundary Poisson) kernel $\mathbb{M} = D_\gamma \mathbb{G}_0$ exists

$$D_\gamma \mathbb{G}_0(\zeta, x) := \lim_{\Omega \ni y \to \zeta} \frac{\mathbb{G}_0(y,x)}{\delta(y)^\gamma} \asymp \frac{\delta(x)^\gamma}{|x-\zeta|^{n-2s+2\gamma}}, \; x \in \Omega, \; \zeta \in \partial\Omega,$$

$$\text{(K1)} \quad \mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma \delta(y)^\gamma}{|x-y|^{2\gamma}} \right)$$

Large $\mathcal{L}$-harmonic functions

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ \delta^b u \asymp \mathcal{M}(1)^{-1} u = h & \text{on } \partial\Omega, \end{cases} \qquad b := \gamma - (2s-1).$$

- Martin operator $\mathcal{M} : L^\infty(\partial\Omega) \longrightarrow \delta^{-b} L^\infty(\Omega)$

$$h \longmapsto \mathcal{M}(h) = \int_{\partial\Omega} \mathbb{M}(\zeta, \cdot) h(\zeta) \, d\mathcal{H}^{n-1}_\zeta$$

  is continuous since

$$\delta(x)^b \mathcal{M}(1)(x) \asymp \int_{\partial\Omega} \frac{\delta(x)^{1-2s+2\gamma}}{|x-\zeta|^{n-2s+2\gamma}} \, d\mathcal{H}^{n-1}_\zeta \asymp \int_{\mathbb{R}^{n-1}} \frac{d\zeta'}{|e_n - \zeta'|^{n-2s+2\gamma}} \asymp 1 \qquad \text{as } x \to \partial\Omega$$

- $\mathcal{L}[\mathcal{M}(h)](x) = 0$ since $\mathbb{M}(\zeta, x) = D_\gamma \mathbb{G}_0(\zeta, x)$
- $\mathcal{M}(1)^{-1}\mathcal{M}(h) \to h$ towards $\partial\Omega$, for $h \in C(\partial\Omega)$

$$\text{(K2)} \quad \exists\, \mathbb{M}(\zeta, x) \asymp \frac{\delta(x)^\gamma}{|x-\zeta|^{n-2s+2\gamma}}, \qquad x \in \Omega,\ \zeta \in \partial\Omega.$$

The large eigenvalue problem

$$\begin{cases} \mathcal{L}v - \lambda v = 0 & \text{in } \Omega, \\ v(x) \to +\infty & \text{as } x \to \partial\Omega. \end{cases}$$

More precisely and generally: $\begin{cases} \mathcal{L}v - \lambda v = g & \text{in } \Omega, \\ \mathcal{M}(1)^{-1}u = h & \text{on } \partial\Omega. \end{cases}$

We will answer:

- Existence and regularity for $\lambda \in \mathbb{R} \setminus \Sigma$?
- Interior and boundary blow-up as $\lambda \to \Sigma$?

Strategy:

- $u = v - \mathcal{M}(h)$ solves $\begin{cases} \mathcal{L}u - \lambda u = g + \lambda\mathcal{M}(h) & \text{in } \Omega, \\ \mathcal{M}(1)^{-1}u = 0 & \text{on } \partial\Omega. \end{cases}$
- Projected $(\Sigma \setminus (0, \lambda))$ linear theory for $\mathcal{G}_\lambda = (\mathcal{L} - \lambda)^{-1}$
- $v = \mathcal{G}_\lambda(g + \lambda\mathcal{M}(h)) + \mathcal{M}(h)$ = explicit projections + errors

Regularity of $\mathcal{G}_0 = \mathcal{L}^{-1}$

- $\mathcal{G}_0 : f \longmapsto \mathcal{G}_0(f) = \int_\Omega \mathbb{G}_0(\cdot, y) f(y)\, dy$ is continuous from

$$
\begin{aligned}
L^1(\Omega; \mathcal{G}_0(\delta^\alpha)) &\longrightarrow L^1(\Omega, \delta^\alpha), && \text{for } \alpha > -1-\gamma,\\
L^1(\Omega) &\longrightarrow L^p(\Omega), && \text{for } p \in [1, \tfrac{n}{n-2s}),\\
L^2(\Omega) &\longrightarrow \mathbb{H}^2_{\mathcal{L}}(\Omega), &&\\
L^{p_0}(\Omega) &\longrightarrow L^{p_1}(\Omega) && \text{for } p_0 \in (1, \tfrac{n}{2s}) \text{ and } \tfrac{1}{p_1} = \tfrac{1}{p_0} - \tfrac{2s}{n},\\
L^q(\Omega) &\longrightarrow L^\infty(\Omega), && \text{for } q \in (\tfrac{n}{2s}, +\infty),\\
\delta^\alpha L^\infty(\Omega) &\longrightarrow \mathcal{G}_0(\delta^\alpha) L^\infty(\Omega), && \text{for } \alpha > -1-\gamma,
\end{aligned}
$$

where (Abatangelo–Gómez-Castro–Vázquez, 2019)

$$
\mathcal{G}_0(\delta^\alpha) \asymp \begin{cases} \delta^{\alpha+2s} & \text{for } \alpha + 2s < \gamma,\\ \delta^\gamma (1 + |\log \delta|) & \text{for } \alpha + 2s = \gamma,\\ \delta^\gamma & \text{for } \alpha + 2s > \gamma. \end{cases}
$$

$$
\text{(K1)} \quad \mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right) \left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)
$$

Eigenfunction estimates

$$\text{(K1)} \quad \mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma}{|x-y|^\gamma}\right) \left(1 \wedge \frac{\delta(y)^\gamma}{|x-y|^\gamma}\right)$$
$$\gtrsim \delta(x)^\gamma \delta(y)^\gamma$$

Behavior of eigenfunctions $\varphi_j = \lambda_j \mathcal{G}_0(\varphi_j)$?

- Upper bound for φ_j, $\forall j \geq 1$

$$\varphi_j = \lambda_j^k \mathcal{G}_0^k(\varphi_j) \in \delta^\gamma L^\infty(\Omega) \qquad \text{(finite large } k\text{)}$$

- Lower bound for φ_1 (Hopf)

$$\varphi_1(x) \gtrsim \lambda_1 \int_\Omega \delta(x)^\gamma \delta(y)^\gamma \varphi_j(y)\, dy \gtrsim \delta(x)^\gamma$$

$\implies$ By duality, our largest space is $L^1(\Omega; \delta^\gamma)$
Related upper bound: $\gamma < 2s$ (RFL, CFL) $\implies \mathcal{G}_0(1) \in \delta^\gamma L^\infty(\Omega)$

Weak-dual solutions

$$\begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega, \\ \mathcal{M}(1)^{-1}u = h & \text{on } \partial\Omega. \end{cases}$$

- All equivalent (if $\exists \mathcal{G}_\lambda$)! (C.–Gómez-Castro–Vázquez, JFA 2021)
  - $\mathcal{G}_0(L_c^\infty(\Omega))$-weak sol. (Abatangelo–Gómez-Castro–Vázquez, 2019) for $f, u \in L^1(\Omega; \delta^\gamma)$, $h \in C(\partial\Omega)$: $\forall \psi \in L_c^\infty(\Omega)$,

$$\int_\Omega u(\psi - \lambda \mathcal{G}_0(\psi))\,dx = \int_\Omega f \mathcal{G}_0(\psi)\,dx + \int_{\partial\Omega} D_\gamma \mathcal{G}_0(\psi)\,h\,d\mathcal{H}^{n-1}.$$

  - $\mathcal{G}_\lambda(\delta^\gamma L^\infty(\Omega))$-weak sol. for $f, u \in L^1(\Omega; \delta^\gamma)$, $h = 0$:

$$\int_\Omega u\psi\,dx = \int_\Omega f \mathcal{G}_\lambda(\psi)\,dx, \qquad \forall \psi \in \delta^\gamma L^\infty(\Omega).$$

  - $\mathcal{G}_0$- or $\mathcal{G}_\lambda$-Green sol. for $f, u \in L^1(\Omega; \delta^\gamma)$, $h = 0$:

$$u - \lambda \mathcal{G}_0(u) = \mathcal{G}_0(f) \quad \text{or} \quad u = \mathcal{G}_\lambda(f).$$

  - Spectral sol. $u \in \mathbb{H}^2_{\mathcal{L}}(\Omega)$ for $f \in L^2(\Omega)$, $h = 0$:

$$(\lambda_j - \lambda)\langle u, \varphi_j \rangle = \langle f, \varphi_j \rangle, \qquad \forall j \geq 1.$$

Existence of $\mathcal{G}_\lambda = (\mathcal{L} - \lambda)^{-1}$ for $\lambda \notin \Sigma$

$$\begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega, \\ \mathcal{M}(1)^{-1} u = 0 & \text{on } \partial\Omega \end{cases} \iff \int_\Omega u\psi \, dx = \int_\Omega f \mathcal{G}_\lambda(\psi) \, dx$$

- $f \in L^2(\Omega) \implies u = \sum_{j \geq 1} \frac{\langle f, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j.$
- $f \in L^{p'}(\Omega)$, $p' > \frac{n}{2s} \implies \|u\|_{L^\infty(\Omega)} \lesssim \|u\|_{L^2(\Omega)} + \|f\|_{L^{p'}(\Omega)}$
- $f \in L^1(\Omega)$ (Brezis, 1971)
  - $u_k = \mathcal{G}_\lambda(f_k)$ $(|f_k| \leq k)$, $\psi = \mathrm{sign}(u_k - u_\ell) \implies u_k \xrightarrow{L^1(\Omega)} u$
  - $\psi = (|u| \wedge k)^{p-1} \mathrm{sign}(u) \implies \|u\|_{L^p(\Omega)} \lesssim \|f\|_{L^1(\Omega)}$, $p < \frac{n}{n-2s}$
- $f \in L^1(\Omega; \delta^\alpha)$
  - $\psi = \mathrm{sign}(u_k - u_\ell)\delta^\alpha \implies \exists u$, $\|u\delta^\alpha\|_{L^1(\Omega)} \lesssim \|f\mathcal{G}_0(\delta^\alpha)\|_{L^1(\Omega)}$

Thus $\mathcal{G}_\lambda$ is defined on $L^1(\Omega; \delta^\gamma)$, same regularity as $\mathcal{G}_0$.

Regularity of of $\mathcal{G}_\lambda = (\mathcal{L} - \lambda)^{-1}$ for $\lambda \notin \Sigma$

- $\mathcal{G}_\lambda$ is well-defined and continuous from

$$
\begin{aligned}
L^1(\Omega; \mathcal{G}_0(\delta^\alpha)) &\longrightarrow L^1(\Omega, \delta^\alpha), && \text{for } \alpha > -1-\gamma, \\
L^1(\Omega) &\longrightarrow L^p(\Omega), && \text{for } p \in [1, \tfrac{n}{n-2s}), \\
L^2(\Omega) &\longrightarrow \mathbb{H}^2_{\mathcal{L}}(\Omega), && \\
L^{p_0}(\Omega) &\longrightarrow L^{p_1}(\Omega) && \text{for } p_0 \in (1, \tfrac{n}{2s}) \text{ and } \tfrac{1}{p_1} = \tfrac{1}{p_0} - \tfrac{2s}{n}, \\
L^q(\Omega) &\longrightarrow L^\infty(\Omega), && \text{for } q \in (\tfrac{n}{2s}, +\infty), \\
\delta^\alpha L^\infty(\Omega) &\longrightarrow \mathcal{G}_0(\delta^\alpha) L^\infty(\Omega), && \text{for } \alpha > -1-\gamma,
\end{aligned}
$$

$$
\begin{aligned}
L^1(\Omega; \delta^\gamma) &\longrightarrow L^p_{\mathrm{loc}}(\Omega), && \text{for } p \in [1, \tfrac{n}{n-2s}), \\
L^1(\Omega; \delta^\gamma) \cap L^{p_0}_{\mathrm{loc}}(\Omega) &\longrightarrow L^{p_1}_{\mathrm{loc}}(\Omega), && \text{for } p_0 \in (1, \tfrac{n}{2s}), \tfrac{1}{p_1} = \tfrac{1}{p_0} - \tfrac{2s}{n}, \\
L^1(\Omega; \delta^\gamma) \cap L^{q_0}_{\mathrm{loc}}(\Omega) &\longrightarrow L^\infty_{\mathrm{loc}}(\Omega), && \text{for } q_0 \in (\tfrac{n}{2s}, +\infty),
\end{aligned}
$$

where
$$
\mathcal{G}_0(\delta^\alpha) \asymp \begin{cases} \delta^{\alpha+2s} & \text{for } \alpha + 2s < \gamma, \\ \delta^\gamma (1 + |\log \delta|) & \text{for } \alpha + 2s = \gamma, \\ \delta^\gamma & \text{for } \alpha + 2s > \gamma. \end{cases}
$$

Projected linear theory

$$\begin{cases} \mathcal{L}u - \lambda u = f & \text{in } \Omega, \\ \mathcal{M}(1)^{-1}u = 0 & \text{on } \partial\Omega \end{cases} \longrightarrow \begin{cases} \mathcal{L}u^\perp - \lambda u^\perp = f^\perp & \text{in } \Omega, \\ \mathcal{M}(1)^{-1}u^\perp = 0 & \text{on } \partial\Omega. \end{cases}$$

- Degeneration as $\lambda \to \lambda_I \in \Sigma$. How? Look at projections.
- Test function space? Need L^2-projections $\langle u, \varphi_j \rangle \varphi_j$.
  - $u \in L^1(\Omega; \delta^\gamma)$ "tested" against $\varphi_j \in \delta^\gamma L^\infty(\Omega)$
  - $\mathcal{G}_\lambda(\delta^\gamma L^\infty(\Omega)) \subset \delta^\gamma L^\infty(\Omega)$ better than $\mathcal{G}_0(L^\infty_c(\Omega))$
  - $(\gamma < 2s \implies \mathcal{G}_\lambda(L^\infty(\Omega)) \subset \delta^\gamma L^\infty(\Omega))$
- Decomposition $u = \sum_{j=1}^\infty \langle u, \varphi_j \rangle \varphi_j$?
  - NOT for $u \in L^1(\Omega; \delta^\gamma)$ unless $u \in L^2(\Omega)$
- "Correct" decomposition:

$$\underbrace{u}_{\in L^1(\Omega;\delta^\gamma)} = \underbrace{\sum_{j=1}^{I} \langle u, \varphi_j \rangle \varphi_j}_{\in E \subset \delta^\gamma L^\infty(\Omega) \subset L^2(\Omega)} + \underbrace{u^\perp}_{\in L^1(\Omega;\delta^\gamma) \cap E^\perp}.$$

L^2-projected (weighted) L^1-theory

$$\begin{cases} \mathcal{L}u^\perp - \lambda u^\perp = f^\perp & \text{in } \Omega, \\ \mathcal{M}(1)^{-1} u = 0 & \text{on } \partial\Omega \end{cases} \iff \int_\Omega u^\perp \psi \, dx = \int_\Omega f^\perp \mathcal{G}_\lambda(\psi) \, dx$$

- C.–Gómez-Castro–Vázquez (JFA 2021): the projection

$$u^\perp = u - \sum_{j=1}^{I+1} \langle u, \varphi_j \rangle \varphi_j \in L^1(\Omega; \delta^\gamma) \cap \{\varphi_1, \ldots, \varphi_I\}^\perp$$

  is controlled by $f^\perp$ uniformly as $\lambda \to \lambda_I$.

- Key: $\int_\Omega u^\perp \psi \, dx = \int_\Omega f^\perp \mathcal{G}_\lambda(\psi)^\perp \, dx = \int_\Omega f^\perp \mathcal{G}_\lambda(\psi^\perp) \, dx$

- Uniform estimates in L^2 (and higher)

$$\mathcal{G}_\lambda(\psi^\perp) = \sum_{j > I+1} \frac{\langle \psi, \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j, \quad \left\| \mathcal{G}_\lambda(\psi^\perp) \right\|_{L^2(\Omega)} \le \frac{\|\psi\|_{L^2(\Omega)}}{\lambda_{I+1} - \lambda_I}$$

- $\psi = \operatorname{sign}(u^\perp) \delta^\alpha \in \delta^\alpha L^\infty(\Omega)$ or $(|u^\perp| \wedge k)^{p-1} \operatorname{sign}(u^\perp) \in L^{p'}(\Omega)$

"Large eigenfunctions"

Theorem (C.–Gómez-Castro–Vázquez, JFA 2021)

$$\begin{cases} \mathcal{L}v - \lambda v = g, & \text{in } \Omega, \\ \delta^{-b} u \asymp \mathcal{M}(1)^{-1} u = h, & \text{on } \partial\Omega, \end{cases} \qquad b = \gamma - (2s-1).$$

$$\text{(K1)}\ \mathbb{G}_0(x,y) \asymp \frac{1}{|x-y|^{n-2s}} \left(1 \wedge \frac{\delta(x)^\gamma \delta(y)^\gamma}{|x-y|^{2\gamma}}\right), \quad \text{(K2)}\ \exists \mathbb{M} = D_\gamma \mathbb{G}_0 \text{ on } \partial\Omega \times \Omega$$

Assume $g \in L^1(\Omega; \delta^\gamma) \cap L^\infty_{\mathrm{loc}}(\Omega)$, $h \in L^\infty(\partial\Omega)$, *(K1), (K2). Then*

- $\forall \lambda \in \mathbb{R} \setminus \Sigma$, $\exists!\, v_\lambda \in L^1(\Omega; \delta^\gamma) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *represented by*

$$v_\lambda = \mathcal{M}(h) + \sum_{j=1}^{I} \frac{\langle g + \lambda \mathcal{M}(h), \varphi_j \rangle}{\lambda_j - \lambda} \varphi_j + \mathcal{P}_{\{\varphi_j\}_{j \le I}^\perp} \big(g + \lambda \mathcal{M}(h)\big),$$

where $\mathcal{M}(h) \in \delta^{-b} L^\infty(\Omega)$, $\mathcal{P}_{\{\varphi_j\}_{j \le I}^\perp} \big(g + \lambda \mathcal{M}(h)\big) \in L^{\frac{n}{n-2s}-} \cap L^\infty_{\mathrm{loc}}(\Omega)$

- *Fredholm alternative on whether* $\langle g + \lambda_j, \mathcal{M}(h), \varphi \rangle = 0$, $\forall (\lambda_j, \varphi)$
- *Total blow-up as* $\lambda \nearrow \lambda_1$ *if* $g \ge 0$, $h > 0$.
- *Convergence as* $s \nearrow 1$ *for RFL, SFL etc.;* $\lim b = 0$.

Evolution equation

Theorem (C.–Gómez-Castro–Vázquez, preprint 2020)

If $\mathcal{L}$ satisfies (K1), (K2), generates a submarkovian semigroup, and $\mathcal{L}^{-1} : \delta^\gamma L^\infty(\Omega) \to \delta^\gamma C(\overline{\Omega})$, then there exists a unique weak-dual solution of

$$\begin{cases} u_t + \mathcal{L}u = f(t,x) & \text{for } x \in \Omega,\ t \in (0,T), \\ \mathcal{M}(1)^{-1}u = h(t,\zeta) & \text{for } \zeta \in \partial\Omega,\ t \in (0,T), \\ u(t,x) = 0 & \text{for } x \in \Omega^c,\ t \in (0,T), \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

for $u_0 \in L^1(\Omega, \delta^\gamma)$, $f \in L^1(0,T; L^1(\Omega,\delta^\gamma))$, $h \in L^1((0,T) \times \partial\Omega)$.

- Turn boundary singularity ON/OFF as you prescribe h !

The end

Thank you very much