1-skeleton of the polytope of pyramidal tours with step-backs

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Definition
The 1-skeleton of a polytope $P$ is the graph whose vertex set is the vertex set of $P$ and edge set is the set of 1-faces of $P$.

Comment
We consider 0/1-polytopes that are associated with combinatorial optimization problems and arise from LP formulations of problems.
Objects of interest (I)

1. Adjacency relation

- It is quite useful for research of 1-skeletons.
- Serves as a neighborhood structure and the basis for edge-following algorithms.
Graph diameter (the maximum edge distance between any pair of vertices)

The diameter serves as the lower bound on the number of steps of the edge-following algorithms like the simplex-method.
Clique number (the number of vertices in the largest clique)

The clique number serves as the lower bound on the number of steps in a special class of direct-type algorithms (some of the branch and bound algorithms, dynamic programming, etc.).
Traveling salesperson problem (TSP)

Asymmetric traveling salesperson problem. Given a complete weighted digraph \( D_n = (V, E) \), it is required to find a Hamiltonian tour of minimum weight.
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Traveling salesperson polytope

We consider a complete graph $D_n$ with the edge set $E$. Let $HT_n$ be the set of all Hamiltonian tours in $D_n$. With each tour $x \in HT_n$ we associate a characteristic vector $x^v \in \mathbb{R}^E$ by the following rule:

$$x^v_e = \begin{cases} 
1, & \text{if an edge } e \text{ is contained in the tour } x, \\
0, & \text{otherwise.} 
\end{cases}$$

The polytope

$$ATSP(n) = \text{conv}\{x^v \mid x \in HT_n\}$$

is called the asymmetric traveling salesperson polytope.

The symmetric traveling salesperson polytope $TSP(n)$ is defined similarly.

$x^v = (1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1)$
What is known?

**Adjacency (Papadimitriou, 1978)**
The question of whether two vertices of the polytopes ATSP\(n\) or TSP\(n\) are nonadjacent is NP-complete.

**Diameter (Padberg, Rao, 1974; Rispoli, Cosares, 1998)**
The diameter of ATSP\(n\) 1-skeleton equals 2 for all \(n \geq 6\).
The diameter of TSP\(n\) 1-skeleton is bounded above by 4.

**Clique number (Bondarenko, 1983)**
The clique number of the ATSP\(n\) 1-skeleton is superpolynomial in dimension:

\[
\omega(\text{ATSP}(n)) \geq 2^{(\sqrt{n-9})/2}.
\]
Special cases
Pyramidal tours

Definition

A Hamiltonian tour is called a *pyramidal* if the salesperson starts in the city 1, then visits some cities in ascending order, reaches the city \( n \), and returns to the city 1 visiting the remaining cities in descending order.


1. A minimum cost pyramidal tour can be determined in \( O(n^2) \) time by dynamic programming.
2. There exists certain combinatorial structures of distance matrices that guarantee the existence of the shortest tour that is pyramidal.
Pyramidal tours with step-backs

Definition (Enomoto, Oda, Ota, 1998)

Let $\tau$ be a Hamiltonian tour. A city $i$ satisfying $\tau^{-1}(i) < i$ and $\tau(i) > i$ is called a peak. A step-back peak is a city $i$ such that

$$\tau^{-1}(i) < i, \tau(i) = i - 1 \text{ and } \tau^2(i) > i, \text{ or } \tau^{-2}(i) > i, \tau^{-1}(i) = i - 1 \text{ and } \tau(i) < i.$$ 

A proper peak is a peak $i$ which is not a step-back peak. A pyramidal tour with step-backs is a Hamiltonian tour $\tau$ which has exactly one proper peak $n$. 
Pyramidal polytopes

We denote by $PT_n$ the set of all pyramidal tours and by $PSBT_n$ the set of all pyramidal tours with step-backs in the complete digraph $D_n = (V, E)$. With each pyramidal tour (with step-backs) $x \in PT_n$ ($x \in PSBT_n$) we associate a characteristic vector $x^v \in \mathbb{R}^E$ by the following rule:

$$x^v_e = \begin{cases} 1, & \text{if an edge } e \in E \text{ is contained in the tour } x, \\ 0, & \text{otherwise.} \end{cases}$$

The polytope

$$PYR(n) = \text{conv}\{x^v \mid x \in PT_n\}$$

is called the polytope of pyramidal tours. The polytope

$$PSB(n) = \text{conv}\{x^v \mid x \in PSBT_n\}$$

is called the polytope of pyramidal tours with step-backs.
1. Vertex adjacency – auxiliary statements

Let $x$ and $y$ be two pyramidal tours with step-backs. We denote by $x^v$ and $y^v$ the corresponding vertices of the $\text{PSB}(n)$ polytope and by $x \cup y$ a regular directed multigraph that contains all edges of both tours $x$ and $y$.

**Sufficient condition for nonadjacency**

Given two tours $x$ and $y$, if the multigraph $x \cup y$ includes a pair of edge-disjoint pyramidal tours with step-backs, different from $x$ and $y$, then the corresponding vertices $x^v$ and $y^v$ of the polytope $\text{PSB}(n)$ are not adjacent.

**Necessary condition for nonadjacency**

If the vertices $x^v$ and $y^v$ of the polytope $\text{PSB}(n)$ are not adjacent, then the multigraph $x \cup y$ includes at least two pyramidal tours with step-backs, different from $x$ and $y$. 
1. Vertex adjacency – a combinatorial problem

**Input:**

\[ x \cup y \]

\[
\begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \leftarrow & 4 & \rightarrow & 5 & \leftarrow & 6 \\
& & & & & & & & & & \\
\end{array}
\]

**Output:**

\[
\begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \leftarrow & 4 & \rightarrow & 5 & \leftarrow & 6 \\
& & & & & & & & & & \\
\end{array}
\]

The corresponding vertices \( x^v \) and \( y^v \) of the polytope PSB(\( n \)) are not adjacent.
Pyramidal encoding

With every pyramidal tour with step-backs $x$ we associate a vector $x^{0,1,\text{sb}}$ of length $n - 2$ by the following rule:

$$x_i^{0,1,\text{sb}} = \begin{cases} 
1, & \text{if } i \text{ is visited by } x \text{ in ascending order}, \\
\overleftarrow{1}, & \text{if } i \text{ is a step-back peak in ascending order}, \\
0, & \text{if } i \text{ is visited by } x \text{ in descending order}, \\
\overrightarrow{0}, & \text{if } i \text{ is a step-back peak in descending order}.
\end{cases}$$

$$x^{0,1,\text{sb}} = \langle 1, 0, \overleftarrow{1}, 1, 0, 1 \rangle$$
1. Vertex adjacency – final criterion

We consider 12 blocks of the following form (a wavy line means that the corresponding coordinate can either contain a step-back or not):

\[
\begin{align*}
U_{11} &= \begin{pmatrix} 1 \ 1 \\ 1 \end{pmatrix}, \\
U_{00} &= \begin{pmatrix} 0 \ 0 \\ 0 \end{pmatrix}, \\
U_{1111} &= \begin{pmatrix} 1 \ 1 \\ 1 \ 1 \end{pmatrix}, \\
U_{0000} &= \begin{pmatrix} 0 \ 0 \\ 0 \ 0 \end{pmatrix}, \\
L_{1110} &= \begin{pmatrix} 1 \ 1 \\ 1 \ 0 \end{pmatrix}, \\
L_{1011} &= \begin{pmatrix} 1 \ 0 \\ 1 \ 1 \end{pmatrix}, \\
L_{0001} &= \begin{pmatrix} 0 \ 0 \\ 0 \ 1 \end{pmatrix}, \\
L_{0100} &= \begin{pmatrix} 0 \ 1 \\ 0 \ 0 \end{pmatrix}, \\
R_{1101} &= \begin{pmatrix} 1 \ 1 \\ 0 \ 1 \end{pmatrix}, \\
R_{0111} &= \begin{pmatrix} 0 \ 1 \\ 1 \ 1 \end{pmatrix}, \\
R_{0010} &= \begin{pmatrix} 0 \ 0 \\ 1 \ 0 \end{pmatrix}, \\
R_{1000} &= \begin{pmatrix} 1 \ 0 \\ 0 \ 0 \end{pmatrix}.
\end{align*}
\]

Adjacency relation criterion

Vertices \( x^v \) and \( y^v \) of the polytope \( \text{PSB}(n) \) (or \( \text{PYR}(n) \)) are not adjacent if and only if the encodings of the tours \( x \) and \( y \) have a left block of the form \( U, L \), have a right block of the form \( U, R \), and some additional conditions are satisfied.
Some examples (I)

\[
\begin{align*}
\text{x} & \langle 1 & 1 & 1 & 1 & 1 & 1 \rangle \\
\text{y} & \langle 1 & 1 & 1 & 1 & 0 \rangle \\
\text{x} & 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \\
\text{y} & 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \\
\text{z} & 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \\
\text{w} & 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7
\end{align*}
\]
Some examples (II)

\[
\begin{align*}
x & \langle 1 & 0 & 0 & 1 & 0 \rangle \\
y & \langle 1 & 0 & 0 & 0 & 0 \rangle \\
\end{align*}
\]

\[
\begin{align*}
x & \quad 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 \quad \\
y & \quad 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 \quad \\
z & \quad 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 \quad \\
w & \quad 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 \quad \\
\end{align*}
\]
**Comment**: vertex adjacency in 1-skeleton of the polytope PSB($n$) (or PYR($n$)) can be verified in linear time $O(n)$.

\[
\begin{align*}
LBlock & \leftarrow \text{TRUE} \\
RBlock, z\text{Diff}_x, z\text{Diff}_y & \leftarrow \text{FALSE} \\
\text{for } i & \leftarrow 2 \text{ to } n - 1 \text{ do} \\
\quad & \text{if } z(x) \text{ is different from } y \text{ then} \\
\quad & \qquad z\text{Diff}_y \leftarrow \text{TRUE} \\
\quad & \text{if } LBlock = \text{FALSE} \text{ then} \\
\quad & \qquad \text{if we found } U \text{ or } L \text{ block then} \\
\quad & \qquad \quad LBlock \leftarrow \text{TRUE} \\
\quad & \text{end if} \\
\text{end if} \\
\text{if } LBlock = \text{TRUE} \text{ and } RBlock = \text{FALSE} \text{ then} \\
\quad & \text{if } z(y) \text{ is different from } x \text{ in the central part then} \\
\quad & \qquad z\text{Diff}_x \leftarrow \text{TRUE} \\
\quad & \text{if } z\text{Diff}_x = \text{TRUE} \text{ and we found } U \text{ or } R \text{ block then} \\
\quad & \qquad RBlock \leftarrow \text{TRUE} \\
\quad & \text{end if} \\
\quad & \text{if } \text{the condition of the central part is violated then} \\
\quad & \qquad LBlock, z\text{Diff}_x \leftarrow \text{FALSE} \\
\text{end if} \\
\text{end for} \\
\text{if } LBlock, z\text{Diff}_x, z\text{Diff}_y = \text{TRUE} \text{ then} \\
\quad & \text{return sufficient condition of nonadjacency is satisfied} \\
\text{else} \\
\quad & \text{return sufficient condition of nonadjacency is not satisfied} \\
\text{end if}
\end{align*}
\]

\begin{itemize}
\item Consider the city 1 as a left block
\item $z$ visits $i$ by the edges of $x$
\item Left part
\item Central part
\item $z$ visits $i$ by the edges of $y$
\item Go to the right part
\item Go to the left part
\item Consider the city $n$ as a right block
\end{itemize}
2. Graph diameter

Theorem (Bondarenko, A.N., 2018)

The diameter of PYR\((n)\) 1-skeleton equals 2 for all \(n \geq 6\).

Theorem

The diameter of the 1-skeleton of the polytope PSB\((n)\) is bounded above by 4.

Idea of proof

- Pyramidal tour \(A'\) with step-backs \(A\)
- Pyramidal tour \(B'\) with all 1 or 0
- Pyramidal tour with step-backs \(B\)
3. Clique number

Theorem

The clique numbers of 1-d skeleta for both polytopes \( \text{PYR}(n) \) and \( \text{PSB}(n) \) are quadratic in the parameter \( n \):

\[
\omega(\text{PYR}(n)) = \omega(\text{PSB}(n)) = \Theta(n^2).
\]

Example of the lower bound

\[
\langle 0 \ 0 \ 0 \ | \ 0 \ 0 \ 0 \rangle \quad \langle 0 \ 0 \ 0 \ | \ 0 \ 0 \ 1 \rangle \quad \langle 0 \ 0 \ 0 \ | \ 0 \ 1 \ 1 \rangle \quad \langle 0 \ 0 \ 0 \ | \ 1 \ 1 \ 1 \rangle \\
\langle 1 \ 0 \ 0 \ | \ 0 \ 0 \ 0 \rangle \quad \langle 1 \ 0 \ 0 \ | \ 0 \ 0 \ 1 \rangle \quad \langle 1 \ 0 \ 0 \ | \ 0 \ 1 \ 1 \rangle \quad \langle 1 \ 0 \ 0 \ | \ 1 \ 1 \ 1 \rangle \\
\langle 1 \ 1 \ 0 \ | \ 0 \ 0 \ 0 \rangle \quad \langle 1 \ 1 \ 0 \ | \ 0 \ 0 \ 1 \rangle \quad \langle 1 \ 1 \ 0 \ | \ 0 \ 1 \ 1 \rangle \quad \langle 1 \ 1 \ 0 \ | \ 1 \ 1 \ 1 \rangle \\
\langle 1 \ 1 \ 1 \ | \ 0 \ 0 \ 0 \rangle \quad \langle 1 \ 1 \ 1 \ | \ 0 \ 0 \ 1 \rangle \quad \langle 1 \ 1 \ 1 \ | \ 0 \ 1 \ 1 \rangle \quad \langle 1 \ 1 \ 1 \ | \ 1 \ 1 \ 1 \rangle
\]
## Conclusion

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**Comment:** complexity of verifying vertex adjacency in 1-skeleton here coincides with the complexity of finding a Hamiltonian decomposition of 4-regular union multigraph $x \cup y$. 
Thank you for your attention!