Infinitesimal Torelli for elliptic surfaces revisited

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Elliptic surfaces

- Today: elliptic surface means compact complex surface with a minimal genus one fibration, without *multiple fibers*.
- No requirement for a section.
- No requirement that the surface is algebraic.
- Let X be an elliptic surface, let C be the base curve for the elliptic fibration $\pi : X \to C$, let g be the genus of C.
- The j-map, sending $p \in C$ to the j-invariant of $\pi^{-1}(p)$, plays an important role in the sequel.

Fundamental line bundle

- Let $\mathcal{L} = (R^1\pi_*\mathcal{O}_X)^*$ (fundamental line bundle).
- Let $d = \deg(\mathcal{L})$.
- If X is not a product then $p_g(X) = \dim H^0(\Omega^2_X) = d + g - 1$.
- If j is constant and different from $0, 1728$ then π has $2d$ fibers of type I_0^*.
- For $j = 0$ and $j = 1728$ and fixed d there are several fiber configurations possible.

Numerical invariants

$d \setminus g$	0	1	≥ 2
0	$E \times \mathbf{P}^1$	Products and nontrivial fiber bundles	
1, $h^0(\mathcal{L}) > 0$	RES	Base locus of $\lvert\Omega_X^2\rvert$ is non empty	
1, $h^0(\mathcal{L}) = 0$	-	-	$\lvert\Omega_X^2\rvert$ is base point free
2	K3	*	*
3,4,5	*	*	*
≥ 6	*	*	*

Infinitesimal Torelli

- Let Y be a smooth compact Kähler manifold of dimension n. Then Y satisfies infinitesimal Torelli on $H^k(Y, \mathbf{C})$ if the differential of the period map on $H^k(Y, \mathbf{C})$ is injective.
- Using Griffiths' transversality, Hodge symmetry etc this equivalent to whether the map

$$\delta_k : H^1(Y, \Theta_Y) \to \bigoplus_{p=0}^{\lfloor (k-1)/2 \rfloor} \mathrm{Hom}(H^p(Y, \Omega_Y^{k-p}), H^{p+1}(Y, \Omega_Y^{k-p-1}))$$

  is injective.
- The map δ_k is injective if and only if δ_{2n-k} is injective. In particular we may assume that $k \leq n$.
- δ_0 is the zero map.

Infinitesimal Torelli

- Recall that $\pi : X \to C$ is a minimal elliptic fibration.
- If X is not a product. Then $H^1(X) \cong H^1(C)$ and δ_1 is not injective
- For the rest of the talk we concentrate on δ_2, i.e., whether

$$H^1(X, \Theta_X) \to \mathrm{Hom}(H^0(X, \Omega^2), H^1(X, \Omega^1))$$

is injective.

Torelli for elliptic surfaces

- Rational elliptic surfaces do not satisfy infinitesimal Torelli. (Case $(g, d) = (0, 1)$.)
- K3 surfaces do satisfy infinitesimal Torelli. (Case $(g, d) = (0, 2)$.)
- Fiber bundles and base will be treated separately. These surfaces have constant j-invariant and may be non-algebraic.

Very old results

- If $g = 0$ then we have $\Omega_X^2 = \pi^* \mathcal{O}_C(d-2)$. Hence if $d > 2$ then Ω_X^2 is divisible in $\mathrm{Pic}(X)$.
- Lieberman-Wilsker-Peters (1977) proved a result for infinitesimal Torelli for manifolds with divisible canonical bundle. They use some Koszul cohomology argument.
- Kii (1978) proved a similar result. He used this to show infinitesimal Torelli if $g = 0$, $d \geq 3$ and the j-invariant is nonconstant.

Saito's PhD thesis

- M.-H. Saito (1983) proved infinitesimal Torelli
  1. if the j-invariant is nonconstant and $(g, d) \neq (0, 1)$,
  2. if the j-invariant is constant but different from 0,1728 and $g = 0, d > 1$,
  3. if the j-invariant is constant but different from 0,1728 and $g > 0, d \geq 3$.
- Saito had partial results for the case of elliptic fiber bundles. (Both counterexamples to infinitesimal Torelli as positive results)

Somewhat recent results I

- In one of the chapters of my PhD thesis I studied elliptic surfaces with $C = \mathbf{P}^1$ and $\rho(X) = h^{1,1}(X)$. It turned out that there exists finitely many positive dimensional families (2004), e.g.,

$$X_{\alpha,\beta,\gamma} : y^2 = x^3 + [t(t-1)(t-\alpha)(t-\beta)(t-\gamma)]^5$$

  is such a family. (There are six fibers of type II^*, $d = 5$.)
- The period map is constant along such a family. In all cases we have $j = 0$, $j = 1728$. This is consistent with Saito's result.

Somewhat recent results II

Theorem

Suppose $g = 0$ and $d > 2$. Then X does not satisfy infinitesimal Torelli if and only if j is constant and π has $d + 1$ singular fibers.

- The number of singular fibers is at least $\lceil \frac{6}{5} d \rceil \geq d + 1$.
- There exits examples with $d + 1$ singular fibers, but only for $d \leq 5$.
- Ikeda (2019) gave a counterexample to infinitesimal Torelli with $g = 1, d = 1$ and nonconstant j-invariant. This contradicts Saito's result.

Saito's proof

- There are several minor issues with Saito's result, most of which can be easily resolved or apply only to $j = 0, 1728$ case.
- In the case of nonconstant j-invariant there is a single issue:
- Saito correctly shows that there is a torsion $\mathcal{T}$ such that X satisfies infinitesimal Torelli if

$$H^0(\Omega^1_C \otimes \mathcal{L}) \otimes H^0(\mathcal{T}) \to H^0(\Omega^1_C \otimes \mathcal{L} \otimes \mathcal{T})$$

  is surjective.
- However, Saito then claims that this map is surjective for any torsion sheaf $\mathcal{T}$.
- If $d = 1$ and $h^0(\mathcal{L}) > 0$ then $\mathcal{L} \cong \mathcal{O}(p)$. If $\mathcal{T}$ is supported at p then the above map is not surjective. This happens in Ikeda's example.
- The case of constant j-invariant is harder to repair.

Alternative approach: Koszul cohomology

Definition

Let Y be a compact complex manifold. Let $\mathcal{F}$ be a coherent analytic sheaf on Y and let $\mathcal{L}$ be an analytic line bundle on Y. Then for any pair of integers (p, q) we define the Koszul cohomology group $K_{p,q}(Y, \mathcal{F}, \mathcal{L})$ as the cohomology of

$$\begin{aligned} H^0(\mathcal{F} \otimes \mathcal{L}^{(q-1)}) \otimes \wedge^{p+1} H^0(\mathcal{L}) &\to H^0(\mathcal{F} \otimes \mathcal{L}^q) \otimes \wedge^p H^0(\mathcal{L}) \to \\ &\to H^0(\mathcal{F} \otimes \mathcal{L}^{(q+1)}) \otimes \wedge^{p-1} H^0(\mathcal{L}). \end{aligned}$$

If $\mathcal{F} = \mathcal{O}_Y$ then one writes $K_{p,q}(Y, \mathcal{L})$ for $K_{p,q}(Y, \mathcal{O}_Y, \mathcal{L})$.

- LWP77 use a dual definition.
- Aim to reprove infinitesimal Torelli, to cover some of the open cases. In particular $j = 0, 1728$.

Green's result

Green in 1984 wrote a paper in which he proposed the use of Koszul cohomology in algebraic geometry and developed a lot of theory. One of his results is:

Theorem

Let Y be a compact Kähler manifold of dimension n. Suppose Ω^n_Y is base point free. Let $p_g = h^0(\Omega^n_Y)$. Then Y satisfies infinitesimal Torelli if and only if $K_{p_g-2,1}(Y, \Omega^{n-1}, \Omega^n) = 0$.

For our elliptic surface X we have that Ω^2_X is base point free if $d > 1$ or $d = 1$ and $h^0(\mathcal{L}) = 0$. In the latter case $g > 1$.

Green's result applied to j nonconstant

Lemma

Let X be an elliptic surface with $d \geq 2$ or $d = 1$ and $h^0(\mathcal{L}) = 0$ such that the j-invariant is nonconstant. Then $K_{p_g-2,1}(X, \Omega^1, \Omega^2) = 0$.

- Using that $\pi_*\Omega^1_X = \Omega^1_C$ we obtain that

$$K_{p_g-2,1}(X, \Omega^1_X, \Omega^2_X) = K_{p_g-2,1}(C, \Omega^1_C, \Omega^1_C \otimes \mathcal{L})$$

- Koszul duality on C yields

$$K_{p_g-2,1}(C, \Omega^1_C, \Omega^1_C \otimes \mathcal{L}) \cong K_{0,1}(C, \mathcal{O}_C, \Omega^1_C \otimes \mathcal{L})^*.$$

- The latter group is (by definition) the cokernel of the multiplication map

$$H^0(\mathcal{O}) \otimes H^0(\Omega^1_C \otimes \mathcal{L}) \to H^0(\Omega^1_C \otimes \mathcal{L})$$

- This map is obviously surjective.
- If j is nonconstant then infinitesimal Torelli holds unless maybe when $d = 1$ and $h^0(\mathcal{L}) > 0$.

j constant

- Suppose now that the j-invariant is constant.
- We exclude now $d = 0$ (fiber bundles, products), $d = 1$ and $h^0(\mathcal{L}) > 0$ (as before) and $(g, d) = (0, 2)$ (K3 surfaces).
- Again we would like to determine whether $K_{p_g-2,1}(X, \Omega^{n-1}, \Omega^n)$ vanishes.
- However, instead of $\pi_*\Omega^1_X = \Omega^1_C$ we have an exact sequence

$$0 \to \Omega^1_C \to \pi_*\Omega^1_X \to \mathcal{L}(-\Delta) \to 0.$$

- Δ is the reduced divisor supported at the discriminant.
- A one page calculation shows that $K_{p_g-2,1}(X, \Omega^1_X, \Omega^2_X)$ vanishes if and only if the multiplication map

$$\mu_\pi : H^0(C, \Omega^1_C \otimes \mathcal{L}^{-1}(\Delta)) \otimes H^0(C, \Omega^1_C \otimes \mathcal{L}) \to H^0(C, (\Omega^1_C)^2(\Delta))$$

is surjective.
- Let $s = \deg(\Delta)$. Then $s \geq d + 1$. The three line bundles have degree $2g - 2 + s - d, 2g - 2 + d, 4g - 4 + s$.

j constant

- Green, Green-Lazarsfeld have a series of results on when

$$H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$$

  is surjective.
- The H^0-lemma of Green yields that for most choices of (s, d) this map is surjective, namely when
  1. $d \geq 3$ and $s \geq d + 2$
  2. $d = 1, 2$ and $s \geq d + 3$.
  3. $d \in \{1, 2\}$, $s = d + 2$ and $h^0(\mathcal{L}^{-2}(\Delta)) = 0$.
- Recall that $s \geq \frac{6}{5}d$. Hence for $1 \leq d \leq 5$ we have $s \geq d + 1$. For $d > 6$ we have $s \geq d + 2$.
- The H^0-lemma is sufficient to cover all cases with $d \geq 6$. (We do not assume that $j \neq 0, 1728$.)
- We obtain stronger results if we replace the H^0-lemma of Green by results of D.C. Butler.

Main Result

Theorem

Let $\pi : X \to C$ *be an elliptic surface with constant* j*-invariant. Let* $d = \deg(\mathcal{L})$ *and* s *the number of singular fibers. Assume that* $d \geq 2$ *or* $d = 1$ *and* $h^0(\mathcal{L}) = 0$.

If one of the following holds

1. $g = 0$ *and* $d = 2$;
2. $s \geq d + 3$;
3. $s = d + 2$ *and* $d \geq 3$.
4. $s = d + 1$; $h^0(\mathcal{L}^{-1}(\Delta)) = 0$; $g \geq 3$ *and* $\mathrm{Cliff}(C) \geq \min\{4 - d, 2\}$. *If* $d \in \{1, 2\}$ *then one of* $\Omega^1_C \otimes \mathcal{L}$, $\Omega^1_C \otimes \mathcal{L}^{-1}(\Delta)$ *is very ample.*
5. $d \in \{1, 2\}$; $s = d + 2$; $h^0(\mathcal{L}^{-2}(\Delta)) = 0$.
6. $d \in \{1, 2\}$; $s = d + 2$; $h^0(\mathcal{L}^{-2}(\Delta)) \neq 0$; $h^0(\mathcal{L}^{-1}(\Delta)) = 0$; $\mathrm{Cliff}(C) \geq 3 - d$.

then X *satisfies infinitesimal Torelli.*

Counterexamples to infinitesimal Torelli

In some cases we manage to show that

$$\mu_\pi : H^0(C, \Omega^1_C \otimes \mathcal{L}^{-1}(\Delta)) \otimes H^0(C, \Omega^1_C \otimes \mathcal{L}) \to H^0(C, (\Omega^1_C)^2(\Delta))$$

is not surjective:

Theorem

Let $\pi : X \to C$ be an elliptic surface with constant j-invariant. Assume that $d \geq 2$ or $d = 1$ and $h^0(\mathcal{L}) = 0$. If $d = 2$ assume that $g(C) > 0$.

1. *If $s = d + 1$ and $h^0(\mathcal{L}^{-1}(\Delta)) > 0$ or*
2. *if $d = 2$, $g = 1$ and $\mathcal{O}_C(\Delta) \cong \mathcal{L}^2$*

then X does satisfy infinitesimal Torelli.

Remaining case: $d = 1$ and $h^0(\mathcal{L}) > 0$

- If $g = 0$ then this corresponds to rational elliptic surfaces. No Torelli.
- If $g = 1$ then we Ikeda's counterexample.
- For $g > 1$ we have little information. Examples with $g > 1$ are rare.
- One can show that to have $d = 1$ and $h^0(\mathcal{L}) > 0$ we need that C is 6-gonal.
- If we want to have nonconstant j-invariant then C is 4-gonal.

Fiber bundle

- Let $\pi : X \to C$ be an elliptic fiber bundle. Then $\mathcal{L}$ is a torsion line bundle of order $1, 2, 3, 4$ or 6.
- Suppose that $\mathcal{L} \ncong \mathcal{O}$. Then we showed that X satisfies infinitesimal Torelli if and only if the multiplication map

$$\mu_\pi : H^0(\Omega^1_C \otimes \mathcal{L}) \otimes H^0(\Omega^1_C \otimes \mathcal{L}^{-1}) \to H^0((\Omega^1_C)^2)$$

  is surjective.
- If $g(C) = 1$ and $\mathcal{L}$ is nontrivial then the LHS is zero and the RHS is nonzero, so no infinitesimal Torelli.
- (Saito:) If $h^1(X)$ is odd and $\mathcal{L} \cong \mathcal{O}$ then X does not satisfy infinitesimal Torelli.
- (Saito:) If $h^1(X)$ is even, C is not hyperelliptic and $\mathcal{L} \cong \mathcal{O}$ then X does satisfy infinitesimal Torelli.

Summary j nonconstant

$d \setminus g$	0	1	≥ 2
1, $h^0(\mathcal{L}) > 0$	-	C	?
1, $h^0(\mathcal{L}) = 0$	X	X	$+$
2	$+$	$+$	$+$
3,4,5	$+$	$+$	$+$
≥ 6	$+$	$+$	$+$

- X=No such surface exist
- $+$=Infinitesimal torelli holds
- -=Infinitesimal torelli does not hold
- C=There are counterexamples, general case open

Summary j constant

$d \setminus g$	0	1	≥ 2
0	-	C/E/?	C/E/?
1, $h^0(\mathcal{L}) > 0$	-	?	?
1, $h^0(\mathcal{L}) = 0$	X	X	C/E/?
2	+	C/E/?	C/E/?
3,4,5	C/E	C/E/?	C/E/?
≥ 6	+	+	+

- X=No such surface exist
- +=Infinitesimal Torelli holds
- -=Infinitesimal Torelli does not hold
- C=There are counterexamples, general case open
- C/E=There are counterexamples and examples no open cases
- C/E/?=There are counterexamples and examples, general case open

Constant j-invariant/Product-quotient surfaces

- Suppose $\pi : X \to C$ is an elliptic surface with constant j-invariant. Then X is a product-quotient surface.
- Suppose for the moment that the j-invariant is zero. Let E be an elliptic curve with $j(E) = 0$ and let ω be the automorphism of order six, which acts by multiplication by $\zeta = \exp(2\pi I/6)$ on $H^{1,0}(E)$.
- There is $\mathbf{Z}/6\mathbf{Z}$ covering of $\tilde{C} \to C$ and an automorphism τ of $\tilde{C}/C$ such that X is birational to

$$(\tilde{C} \times E)/\langle(\tau, \omega)\rangle$$

- For $j = 1728$ we have an automorphism of order 4 on E and $\mathbf{Z}/4\mathbf{Z}$-cover. For the other j-values we have an automorphism of order 2 and a double cover.

Constant j-invariant/Product-quotient surfaces

- Continue with $j = 0$.
- We can decompose $H^2(X, \mathbf{Q})$ in

$$H^2(\tilde{C} \times E)^{\langle(\tau,\omega)\rangle} \oplus V$$

with $V = \mathbf{C}(-1)^r$.
- We have that that $(2,0)$-part of $H^2(C \times E)^{\langle(\tau,\omega)\rangle}$ equals

$$H^{1,0}(E) \otimes H^{1,0}(\tilde{C})_{\zeta^5}$$

Constant j-invariant/Product-quotient surfaces

- The $(1,1)$ part equals

$$\left(H^{1,0}(E)\otimes H^{0,1}(\tilde{C})_{\zeta^5}\right) \oplus \left(H^{0,1}(E)\otimes H^{1,0}(\tilde{C})_{\zeta}\right)$$
$$\oplus \langle c_1(\tilde{C}\times\{p\}), c_1(\{p\}\times E)\rangle$$

- In the examples of [Klo04] ($g=0$) one has $H^{0,1}(\tilde{C})_{\zeta^5} = H^{1,0}(\tilde{C})_{\zeta} = 0$, which is an obstruction to have a variation of Hodge structures.
- We were not able to pursue this approach in the case $g>0$.