Infinitesimal Torelli for elliptic surfaces revisited

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Elliptic surfaces

- Today: elliptic surface means compact complex surface with a minimal genus one fibration, without *multiple fibers*.
- No requirement for a section.
- No requirement that the surface is algebraic.
- Let $X$ be an elliptic surface, let $C$ be the base curve for the elliptic fibration $\pi : X \to C$, let $g$ be the genus of $C$.
- The $j$-map, sending $p \in C$ to the $j$-invariant of $\pi^{-1}(p)$, plays an important role in the sequel.
Fundamental line bundle

- Let $\mathcal{L} = (R^1\pi_* \mathcal{O}_X)^*$ (fundamental line bundle).
- Let $d = \deg(\mathcal{L})$.
- If $X$ is not a product then $p_g(X) = \dim H^0(\Omega^2_X) = d + g - 1$.
- If $j$ is constant and different from 0, 1728 then $\pi$ has $2d$ fibers of type $I_0^*$.
- For $j = 0$ and $j = 1728$ and fixed $d$ there are several fiber configurations possible.
### Numerical invariants

<table>
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<tr>
<th>$d \setminus g$</th>
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<tr>
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<td>$E \times \mathbb{P}^1$</td>
<td>RES</td>
<td>Products and nontrivial fiber bundles</td>
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<td>1, $h^0(\mathcal{L}) &gt; 0$</td>
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<td>Base locus of $</td>
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<tr>
<td>1, $h^0(\mathcal{L}) = 0$</td>
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Infinitesimal Torelli

Let $Y$ be a smooth compact Kähler manifold of dimension $n$. Then $Y$ satisfies infinitesimal Torelli on $H^k(Y, \mathbb{C})$ if the differential of the period map on $H^k(Y, \mathbb{C})$ is injective.

Using Griffiths’ transversality, Hodge symmetry etc this equivalent to whether the map

$$
\delta_k : H^1(Y, \Theta_Y) \rightarrow \bigoplus_{p=0}^{\lfloor (k-1)/2 \rfloor} \text{Hom}(H^p(Y, \Omega^{k-p}_Y), H^{p+1}(Y, \Omega^{k-p-1}_Y))
$$

is injective.

The map $\delta_k$ is injective if and only if $\delta_{2n-k}$ is injective. In particular we may assume that $k \leq n$.

$\delta_0$ is the zero map.
Recall that \( \pi : X \to C \) is a minimal elliptic fibration.

If \( X \) is not a product. Then \( H^1(X) \cong H^1(C) \) and \( \delta_1 \) is not injective.

For the rest of the talk we concentrate on \( \delta_2 \), i.e., whether

\[
H^1(X, \Theta_X) \to \text{Hom}(H^0(X, \Omega^2), H^1(X, \Omega^1))
\]

is injective.
Torelli for elliptic surfaces

- Rational elliptic surfaces do not satisfy infinitesimal Torelli. (Case \((g, d) = (0, 1)\).)
- K3 surfaces do satisfy infinitesimal Torelli. (Case \((g, d) = (0, 2)\).)
- Fiber bundles and base will be treated separately. These surfaces have constant \(j\)-invariant and may be non-algebraic.
Very old results

- If $g = 0$ then we have $\Omega_X^2 = \pi^* \mathcal{O}_C(d - 2)$. Hence if $d > 2$ then $\Omega_X^2$ is divisible in $\text{Pic}(X)$.

- Lieberman-Wilsker-Peters (1977) proved a result for infinitesimal Torelli for manifolds with divisible canonical bundle. They use some Koszul cohomology argument.

- Kii (1978) proved a similar result. He used this to show infinitesimal Torelli if $g = 0$, $d \geq 3$ and the $j$-invariant is nonconstant.
M.-H. Saito (1983) proved infinitesimal Torelli
1. if the $j$-invariant is nonconstant and $(g, d) \neq (0, 1),$
2. if the $j$-invariant is constant but different from $0, 1728$ and $g = 0, d > 1,$
3. if the $j$-invariant is constant but different from $0, 1728$ and $g > 0, d \geq 3.$

Saito had partial results for the case of elliptic fiber bundles. (Both counterexamples to infinitesimal Torelli as positive results)
In one of the chapters of my PhD thesis I studied elliptic surfaces with $C = \mathbb{P}^1$ and $\rho(X) = h^{1,1}(X)$. It turned out that there exists finitely many positive dimensional families (2004), e.g.,

$$X_{\alpha,\beta,\gamma} : y^2 = x^3 + [t(t-1)(t-\alpha)(t-\beta)(t-\gamma)]^5$$

is such a family. (There are six fibers of type $II^*$, $d = 5$.)

The period map is constant along such a family. In all cases we have $j = 0$, $j = 1728$. This is consistent with Saito’s result.
Theorem

Suppose \( g = 0 \) and \( d > 2 \). Then \( X \) does not satisfy infinitesimal Torelli if and only if \( j \) is constant and \( \pi \) has \( d + 1 \) singular fibers.

- The number of singular fibers is at least \( \lceil \frac{6}{5} d \rceil \geq d + 1 \).
- There exits examples with \( d + 1 \) singular fibers, but only for \( d \leq 5 \).
- Ikeda (2019) gave a counterexample to infinitesimal Torelli with \( g = 1, d = 1 \) and nonconstant \( j \)-invariant. This contradicts Saito’s result.
Saito’s proof

- There are several minor issues with Saito’s result, most of which can be easily resolved or apply only to $j = 0, 1728$ case.
- In the case of nonconstant $j$-invariant there is a single issue:
- Saito correctly shows that there is a torsion $\mathcal{T}$ such that $X$ satisfies infinitesimal Torelli if

$$H^0(\Omega^1_C \otimes L) \otimes H^0(\mathcal{T}) \to H^0(\Omega^1_C \otimes L \otimes \mathcal{T})$$

is surjective.
- However, Saito then claims that this map is surjective for any torsion sheaf $\mathcal{T}$.
- If $d = 1$ and $h^0(L) > 0$ then $L \cong \mathcal{O}(p)$. If $\mathcal{T}$ is supported at $p$ then the above map is not surjective. This happens in Ikeda’s example.
- The case of constant $j$-invariant is harder to repair.
Alternative approach: Koszul cohomology

Definition
Let $Y$ be a compact complex manifold. Let $\mathcal{F}$ be a coherent analytic sheaf on $Y$ and let $\mathcal{L}$ be an analytic line bundle on $Y$. Then for any pair of integers $(p, q)$ we define the Koszul cohomology group $K_{p,q}(Y, \mathcal{F}, \mathcal{L})$ as the cohomology of

$$H^0(\mathcal{F} \otimes \mathcal{L}^{(q-1)}) \otimes \wedge^{p+1} H^0(\mathcal{L}) \rightarrow H^0(\mathcal{F} \otimes \mathcal{L}^q) \otimes \wedge^p H^0(\mathcal{L}) \rightarrow H^0(\mathcal{F} \otimes \mathcal{L}^{(q+1)}) \otimes \wedge^{p-1} H^0(\mathcal{L}).$$

If $\mathcal{F} = \mathcal{O}_Y$ then one writes $K_{p,q}(Y, \mathcal{L})$ for $K_{p,q}(Y, \mathcal{O}_Y, \mathcal{L})$.

- LWP77 use a dual definition.
- Aim to reprove infinitesimal Torelli, to cover some of the open cases. In particular $j = 0, 1728.$
Green’s result

Green in 1984 wrote a paper in which he proposed the use of Koszul cohomology in algebraic geometry and developed a lot of theory. One of his results is:

Theorem

Let $Y$ be a compact Kähler manifold of dimension $n$. Suppose $\Omega^n_Y$ is base point free. Let $p_g = h^0(\Omega^n_Y)$. Then $Y$ satisfies infinitesimal Torelli if and only if $K_{p_g-2,1}(Y, \Omega^{n-1}, \Omega^n) = 0$.

For our elliptic surface $X$ we have that $\Omega^2_X$ is base point free if $d > 1$ or $d = 1$ and $h^0(\mathcal{L}) = 0$. In the latter case $g > 1$. 

Green’s result applied to $j$ nonconstant

**Lemma**

Let $X$ be an elliptic surface with $d \geq 2$ or $d = 1$ and $h^0(\mathcal{L}) = 0$ such that the $j$-invariant is nonconstant. Then $K_{pg-2,1}(X, \Omega^1, \Omega^2) = 0$.

- Using that $\pi_* \Omega^1_X = \Omega^1_C$ we obtain that $K_{pg-2,1}(X, \Omega^1_X, \Omega^2_X) = K_{pg-2,1}(C, \Omega^1_C, \Omega^1_C \otimes \mathcal{L})$.

- Koszul duality on $C$ yields $K_{pg-2,1}(C, \Omega^1_C, \Omega^1_C \otimes \mathcal{L}) \cong K_{0,1}(C, \mathcal{O}_C, \Omega^1_C \otimes \mathcal{L})^*.$

- The latter group is (by definition) the cokernel of the multiplication map $H^0(\mathcal{O}) \otimes H^0(\Omega^1_C \otimes \mathcal{L}) \rightarrow H^0(\Omega^1_C \otimes \mathcal{L})$.

- This map is obviously surjective.

- If $j$ is nonconstant then infinitesimal Torelli holds unless maybe when $d = 1$ and $h^0(\mathcal{L}) > 0$. 
Suppose now that the $j$-invariant is constant.

We exclude now $d = 0$ (fiber bundles, products), $d = 1$ and $h^0(\mathcal{L}) > 0$ (as before) and $(g, d) = (0, 2)$ (K3 surfaces).

Again we would like to determine whether $K_{pg-2,1}(X, \Omega^{n-1}, \Omega^n)$ vanishes.

However, instead of $\pi_*\Omega^1_X = \Omega^1_C$ we have an exact sequence

\[
0 \rightarrow \Omega^1_C \rightarrow \pi_*\Omega^1_X \rightarrow \mathcal{L}(-\Delta) \rightarrow 0.
\]

$\Delta$ is the reduced divisor supported at the discriminant.

A one page calculation shows that $K_{pg-2,1}(X, \Omega^1_X, \Omega^2_X)$ vanishes if and only if the multiplication map

\[
\mu_\pi : H^0(C, \Omega^1_C \otimes \mathcal{L}^{-1}(\Delta)) \otimes H^0(C, \Omega^1_C \otimes \mathcal{L}) \rightarrow H^0(C, (\Omega^1_C)^2(\Delta))
\]

is surjective.

Let $s = \text{deg}(\Delta)$. Then $s \geq d + 1$. The three line bundles have degree $2g - 2 + s - d, 2g - 2 + d, 4g - 4 + s$. 
Green, Green-Lazarsfeld have a series of results on when

$$H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$$

is surjective.

The $H^0$-lemma of Green yields that for most choices of $(s, d)$ this map is surjective, namely when

1. $d \geq 3$ and $s \geq d + 2$
2. $d = 1, 2$ and $s \geq d + 3$.
3. $d \in \{1, 2\}$, $s = d + 2$ and $h^0(\mathcal{L}^{-2}(\Delta)) = 0$.

Recall that $s \geq \frac{6}{5}d$. Hence for $1 \leq d \leq 5$ we have $s \geq d + 1$. For $d > 6$ we have $s \geq d + 2$.

The $H^0$-lemma is sufficient to cover all cases with $d \geq 6$. (We do not assume that $j \neq 0, 1728$.)

We obtain stronger results if we replace the $H^0$-lemma of Green by results of D.C. Butler.
Main Result

Theorem
Let $\pi : X \to C$ be an elliptic surface with constant $j$-invariant. Let $d = \deg(\mathcal{L})$ and $s$ the number of singular fibers. Assume that $d \geq 2$ or $d = 1$ and $h^0(\mathcal{L}) = 0$.
If one of the following holds

1. $g = 0$ and $d = 2$;
2. $s \geq d + 3$;
3. $s = d + 2$ and $d \geq 3$.
4. $s = d + 1$; $h^0(\mathcal{L}^{-1}(\Delta)) = 0$; $g \geq 3$ and $\text{Cliff}(C) \geq \min\{4 - d, 2\}$. If $d \in \{1, 2\}$ then one of $\Omega^1_C \otimes \mathcal{L}$, $\Omega^1_C \otimes \mathcal{L}^{-1}(\Delta)$ is very ample.
5. $d \in \{1, 2\}$; $s = d + 2$; $h^0(\mathcal{L}^{-2}(\Delta)) = 0$.
6. $d \in \{1, 2\}$; $s = d + 2$; $h^0(\mathcal{L}^{-2}(\Delta)) \neq 0$; $h^0(\mathcal{L}^{-1}(\Delta)) = 0$; $\text{Cliff}(C) \geq 3 - d$.

then $X$ satisfies infinitesimal Torelli.
Counterexamples to infinitesimal Torelli

In some cases we manage to show that

$$\mu_\pi : H^0(C, \Omega_C^1 \otimes \mathcal{L}^{-1}(\Delta)) \otimes H^0(C, \Omega_C^1 \otimes \mathcal{L}) \to H^0(C, (\Omega_C^1)^2(\Delta))$$

is not surjective:

**Theorem**

Let $\pi : X \to C$ be an elliptic surface with constant $j$-invariant. Assume that $d \geq 2$ or $d = 1$ and $h^0(\mathcal{L}) = 0$. If $d = 2$ assume that $g(C) > 0$.

1. If $s = d + 1$ and $h^0(\mathcal{L}^{-1}(\Delta)) > 0$ or
2. if $d = 2$, $g = 1$ and $\mathcal{O}_C(\Delta) \cong \mathcal{L}^2$

then $X$ does satisfy infinitesimal Torelli.
Remaining case: $d = 1$ and $h^0(\mathcal{L}) > 0$

- If $g = 0$ then this corresponds to rational elliptic surfaces. No Torelli.
- If $g = 1$ then we Ikeda’s counterexample.
- For $g > 1$ we have little information. Examples with $g > 1$ are rare.
- One can show that to have $d = 1$ and $h^0(\mathcal{L}) > 0$ we need that $C$ is 6-gonal.
- If we want to have nonconstant $j$-invariant then $C$ is 4-gonal.
Fiber bundle

- Let $\pi : X \to C$ be an elliptic fiber bundle. Then $\mathcal{L}$ is a torsion line bundle of order 1, 2, 3, 4 or 6.
- Suppose that $\mathcal{L} \not\cong \mathcal{O}$. Then we showed that $X$ satisfies infinitesimal Torelli if and only if the multiplication map

$$
\mu_\pi : H^0(\Omega^1_C \otimes \mathcal{L}) \otimes H^0(\Omega^1_C \otimes \mathcal{L}^{-1}) \to H^0((\Omega^1_C)^2)
$$

is surjective.
- If $g(C) = 1$ and $\mathcal{L}$ is nontrivial then the LHS is zero and the RHS is nonzero, so no infinitesimal Torelli.
- (Saito:) If $h^1(X)$ is odd and $\mathcal{L} \cong \mathcal{O}$ then $X$ does not satisfy infinitesimal Torelli.
- (Saito:) If $h^1(X)$ is even, $C$ is not hyperelliptic and $\mathcal{L} \cong \mathcal{O}$ then $X$ does satisfy infinitesimal Torelli.
### Summary \( j \) nonconstant

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<tr>
<th>( d ) ( \setminus ) ( g )</th>
<th>0</th>
<th>1</th>
<th>( \geq 2 )</th>
</tr>
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<tbody>
<tr>
<td>1, ( h^0(\mathcal{L}) &gt; 0 )</td>
<td>-</td>
<td>C</td>
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<td>1, ( h^0(\mathcal{L}) = 0 )</td>
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<td>3,4,5</td>
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<td>+</td>
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- **X** = No such surface exist
- **+** = Infinitesimal torelli holds
- **-** = Infinitesimal torelli does not hold
- **C** = There are counterexamples, general case open
### Summary $j$ constant

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<th>0</th>
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<tr>
<td>0</td>
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<td>C/E/?</td>
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- X: No such surface exist
- +: Infinitesimal Torelli holds
- -: Infinitesimal Torelli does not hold
- C: There are counterexamples, general case open
- C/E: There are counterexamples and examples, no open cases
- C/E/?: There are counterexamples and examples, general case open
Constant $j$-invariant/Product-quotient surfaces

- Suppose $\pi : X \to C$ is an elliptic surface with constant $j$-invariant. Then $X$ is a product-quotient surface.
- Suppose for the moment that the $j$-invariant is zero. Let $E$ be an elliptic curve with $j(E) = 0$ and let $\omega$ be the automorphism of order six, which acts by multiplication by $\zeta = \exp(2\pi i/6)$ on $H^{1,0}(E)$.
- There is $\mathbb{Z}/6\mathbb{Z}$ covering of $\tilde{C} \to C$ and an automorphism $\tau$ of $\tilde{C}/C$ such that $X$ is birational to

  $$(\tilde{C} \times E)/\langle(\tau, \omega)\rangle$$

- For $j = 1728$ we have an automorphism of order 4 on $E$ and $\mathbb{Z}/4\mathbb{Z}$-cover. For the other $j$-values we have an automorphism of order 2 and a double cover.
Constant $j$-invariant/Product-quotient surfaces

- Continue with $j = 0$.
- We can decompose $H^2(X, \mathbb{Q})$ in
  \[ H^2(\tilde{C} \times E)^\langle(\tau,\omega)\rangle \oplus V \]
  with $V = \mathbb{C}(-1)^r$.
- We have that that $(2, 0)$-part of $H^2(C \times E)^\langle(\tau,\omega)\rangle$ equals
  \[ H^{1,0}(E) \otimes H^{1,0}(\tilde{C})\zeta^5 \]
Constant $j$-invariant/Product-quotient surfaces

- The $(1, 1)$ part equals

\[
\left( H^{1,0}(E) \otimes H^{0,1}(\tilde{C})_{\zeta^5} \right) \oplus \left( H^{0,1}(E) \otimes H^{1,0}(\tilde{C})_{\zeta} \right) \\
\oplus \langle c_1(\tilde{C} \times \{p\}), c_1(\{p\} \times E) \rangle
\]

- In the examples of [Klo04] ($g = 0$) one has $H^{0,1}(\tilde{C})_{\zeta^5} = H^{1,0}(\tilde{C})_{\zeta} = 0$, which is an obstruction to have a variation of Hodge structures.

- We were not able to pursue this approach in the case $g > 0$. 