Linear functions preserving Green's relations over fields

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A. Guterman (Lomonosov Moscow State University),
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1. A. Guterman, M. Johnson, M. Kambites.
Linear isomorphisms preserving Green's relations for matrices over anti-negative semifields, Linear Algebra Appl. 545 (2018) 1-14.
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Linear functions preserving Green's relations over fields, Linear Algebra Appl. 611 (2021), 310-333.

Green's relations

Let $\mathcal{M}$ be a monoid, $a, b \in \mathcal{M}$. We say that:

Definition

(i) $a \,\mathcal{R}\, b$ if $a\mathcal{M} = b\mathcal{M}$;

(ii) $a \,\mathcal{L}\, b$ if $\mathcal{M}a = \mathcal{M}b$;

(iii) $a \,\mathcal{J}\, b$ if $\mathcal{M}a\mathcal{M} = \mathcal{M}b\mathcal{M}$;

(iv) $a \,\mathcal{H}\, b$ if $a \,\mathcal{R}\, b$ and $a \,\mathcal{L}\, b$;

(v) $a \,\mathcal{D}\, b$ if $\exists\, c \in \mathcal{M}$: $a \,\mathcal{R}\, c$ and $c \,\mathcal{L}\, b$.

Simple properties:

- $\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ are eqiuvalence relations on $\mathcal{M}$
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$
- $\mathcal{H} \subseteq \mathcal{R}, \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$

Square matrices

Let K be a field. Consider the monoid $\mathcal{M} = (M_n(K), \cdot\ )$ of square matrices of order n.

Theorem

For $A, B \in M_n(K)$, it holds that:

(i) $A \,\mathcal{L}\, B \Leftrightarrow \mathrm{Row}_K(A) = \mathrm{Row}_K(B) \Leftrightarrow \mathrm{Ker}\, A = \mathrm{Ker}\, B$;

(ii) $A \,\mathcal{R}\, B \Leftrightarrow \mathrm{Col}_K(A) = \mathrm{Col}_K(B) \Leftrightarrow \mathrm{Im}\, A = \mathrm{Im}\, B$;

(iii) $A \,\mathcal{J}\, B \Leftrightarrow \mathrm{rk}\, A = \mathrm{rk}\, B$;

(iv) $A \,\mathcal{H}\, B \Leftrightarrow \mathrm{Row}_K(A) = \mathrm{Row}_K(B)$ and $\mathrm{Col}_K(A) = \mathrm{Col}_K(B)$;

(v) $A \,\mathcal{D}\, B \Leftrightarrow A \,\mathcal{J}\, B$, i. e., $\mathcal{D} = \mathcal{J}$.

Linear maps

Definition

A linear map $T\colon M_n(K) \to M_n(K)$ preserves a relation $\mathcal{X}$ (on $M_n(K)$) if $A \,\mathcal{X}\, B \Rightarrow T(A) \,\mathcal{X}\, T(B)$ for all $A, B \in M_n(K)$.

Which of the linear maps preserve Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D}$?

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Which of the linear maps preserve Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D}$?

- $T \equiv 0$
- $T(A) = PAQ$, where $P, Q \in GL_n(K)$. Preserves $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D}$
- $T(A) = A^T$. Preserves $\mathcal{H}, \mathcal{J}, \mathcal{D}$
- $T(A) = AX$, where $X \in M_n(K)$. Preserves $\mathcal{L}$
  ($\operatorname{Ker} A = \operatorname{Ker} B \;\Rightarrow\; \operatorname{Ker} AX = \operatorname{Ker} BX$)
- $T(A) = XA$, where $X \in M_n(K)$. Preserves $\mathcal{R}$
- Compositions of the above transformations

Results for semifields

Let S be a semifield:

$\mathbb{R}_+$, $\mathbb{B}$ (boolean semiring), $\langle \mathbb{R} \cup \{-\infty\}, \max, +\rangle$ (tropical semifield), ...

A semifield is either anti-negative (any element except 0 does not have an additive inverse) or a field.

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Theorem (Guterman, Johnson, Kambites, 2018)

Let S be a semifield which is not a field and $T\colon M_n(S) \to M_n(S)$ a bijective S-linear map. The following are equivalent:

(i) T preserves $\mathcal{L}$;

(ii) T preserves $\mathcal{R}$;

(iii) There exist invertible (monomial) matrices $P, Q \in M_n(S)$ such that $T(A) = PAQ$ for all $A \in M_n(S)$.

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(ii) T preserves $\mathcal{D}$;

(iii) T preserves $\mathcal{H}$;

(iv) There exist invertible (monomial) matrices $P, Q \in M_n(S)$ such that $T(A) = PAQ$ for all $A \in M_n(S)$ or $T(A) = PA^TQ$ for all $A \in M_n(S)$.

Relations $\mathcal{J} = \mathcal{D}$, and also $\mathcal{L}, \mathcal{R}$

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let K be a field and $T\colon M_n(K) \to M_n(K)$ a linear map. The following are equivalent:

(i) T preserves $\mathcal{J}$;

(ii) Either $T \equiv 0$, or there exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PA^TQ$ for all $A \in M_n(K)$.

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(iii) There exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$.

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $n \geq 1$ and K be a field in which every polynomial of degree exactly n has a root. The linear maps preserving $\mathcal{L}$ on $M_n(K)$ (resp. $\mathcal{R}$) are precisely those of the form $A \mapsto PAX$ (resp. $A \mapsto XAP$), where $P \in \mathrm{GL}_n(K)$ and $X \in M_n(K)$.

Proof tools

Theorem (Marcus and Moyls 1959; Minc 1977)

Let K be an algebraically closed field and $T\colon M_n(K) \to M_n(K)$ a linear map preserving the set of rank-1 matrices. Then $\exists\, P, Q \in GL_n(K)$:

- $T(A) = PAQ$ for all $A \in M_n(K)$ or
- $T(A) = PA^TQ$ for all $A \in M_n(K)$.

Theorem (Lautemann 1981)

Let K be an arbitrary field and $T\colon M_n(K) \to M_n(K)$ a bijective linear map preserving the set of rank-1 matrices. Then $\exists\, P, Q \in GL_n(K)$:

- $T(A) = PAQ$ for all $A \in M_n(K)$ or
- $T(A) = PA^TQ$ for all $A \in M_n(K)$.

Examples

Statement

For $K = \mathbb{Q}$ and every $n \geq 2$, there exist non-bijective linear $\mathcal{L}$-preservers that do not fit the conditions of the above theorems.

Example (Botta, 1978)

Let $f(x) \in K[x]$ be an irreducible polynomial of degree $n \geq 2$ (if exists). Let C be any matrix in $M_n(K)$ whose minimal polynomial is $f(x)$. Then

$$\det(\lambda_1 I + \lambda_2 C + \lambda_3 C^2 + \ldots + \lambda_n C^{n-1}) = 0 \iff \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0.$$

Hence the linear transformation T defined as $T(A) = \sum_{i=1}^{n} a_{i1} C^{i-1}$, where $A = (a_{ij}) \in M_n(K)$, preserves $\mathcal{L}$ and $\mathcal{H}$, and is not of the form mentioned in the above theorems.

Examples

Statement

If $K = \mathbb{R}$ and n is even, then there exist non-bijective linear $\mathcal{L}$-preservers that do not fit the conditions of the above theorems.

Example (based on a construction of Petrović, 2009)

Let n be even, $n = 2m$.
$C_{2k-1} = E_{2k-1,1} + E_{2k,2}$ and $C_{2k} = E_{2k,1} - E_{2k-1,2}, \quad k = 1, \ldots, \frac{n}{2}$.
$T : M_n(K) \to M_n(K)$ is given by

$$T(A) = \sum_{i=1}^{n} a_{i1} C_i = \begin{pmatrix} a_{1,1} & -a_{2,1} & 0 & \cdots & 0 \\ a_{2,1} & a_{1,1} & 0 & \cdots & 0 \\ a_{3,1} & -a_{4,1} & 0 & \cdots & 0 \\ a_{4,1} & a_{3,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2m-1,1} & -a_{2m,1} & 0 & \cdots & 0 \\ a_{2m,1} & a_{2m-1,1} & 0 & \cdots & 0 \end{pmatrix}$$

$\mathcal{H}$-relation

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let K be a field and $T\colon M_n(K) \to M_n(K)$ a bijective linear map. The following are equivalent:

(i) T preserves $\mathcal{H}$;

(ii) T preserves $\mathcal{J}$;

(iii) There exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PA^TQ$ for all $A \in M_n(K)$.

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $n \geq 1$ and K be a field in which every polynomial of degree exactly n has a root. Then for a linear map T preserving the $\mathcal{H}$-relation on $M_n(K)$, it holds that either $T \equiv 0$, or there exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PA^TQ$ for all $A \in M_n(K)$.

$\mathcal{H}$-relation

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let K be a field and $T\colon M_n(K) \to M_n(K)$ a linear map.
Then T preserves $\mathcal{H} \Leftrightarrow T = 0$ or T preservers the set of non-singular matrices.

Theorem (de Seguins Pazzis, 2010)

Let $n \geq 2$, K be any field, and $T\colon M_n(K) \to M_n(K)$ be a linear non-singularity preserver. Then:
(i) either T is bijective and then there exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PA^TQ$ for all $A \in M_n(K)$;
(ii) or there exist an n-dimensional subspace $V \subset M_n(K)$ contained in $GL_n(K) \cup \{0_{n\times n}\}$, an isomorphism $\alpha\colon K^n \to V$, and a non-zero $x \in K^n$ such that

$$T(M) = \alpha(Mx^T)\ \forall M \in M_n(K) \ \text{ or } \ T(M) = \alpha(M^Tx^T)\ \forall M \in M_n(K)$$

Examples over $\mathbb{R}$

There exist non-bijective maps preserving $\mathcal{H}$ (and also $\mathcal{L}$).

Example

Let $T\colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be defined as follows:

$$T\begin{pmatrix} a & * \\ b & * \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This is the matrix representation of complex numbers.

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This is the matrix representation of complex numbers.

Example

Let $T\colon M_4(\mathbb{R}) \to M_4(\mathbb{R})$ be defined as follows:

$$T\begin{pmatrix} a & * & * & * \\ b & * & * & * \\ c & * & * & * \\ d & * & * & * \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} = aP_1 + bP_2 + cP_3 + dP_4,$$

where $P_1 = I_4$,

$$P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},\ P_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ P_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

This is the matrix representation of quaternions.

(1, 2, 4, 8)-theorem

Theorem (Kervaire, Milnor–Bott, 1958)

Any finite dimensional division algebra over the real numbers has dimension $1, 2, 4$ or 8.

Corollary

If $n \notin \{2, 4, 8\}$, then every linear $\mathcal{H}$-preserver on $M_n(\mathbb{R})$ is either zero or bijective.

Thanks for attention!