Linear functions preserving Green’s relations over fields

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M. Kambites (University of Manchester).

1. A. Guterman, M. Johnson, M. Kambites.
   Linear isomorphisms preserving Green’s relations for matrices over

2. A. Guterman, M. Johnson, M. Kambites, A. Maksaev.
   Linear functions preserving Green’s relations over fields, Linear
   Algebra Appl. 611 (2021), 310-333.
Let $\mathcal{M}$ be a monoid, $a, b \in \mathcal{M}$. We say that:

**Definition**

(i) $a R b$ if $a\mathcal{M} = b\mathcal{M}$;

(ii) $a L b$ if $\mathcal{M}a = \mathcal{M}b$;

(iii) $a J b$ if $\mathcal{M}a\mathcal{M} = \mathcal{M}b\mathcal{M}$;

(iv) $a H b$ if $a R b$ and $a L b$;

(v) $a D b$ if $\exists c \in \mathcal{M}$: $a R c$ and $c L b$.

Simple properties:

- $R, L, J, H, D$ are equivalence relations on $\mathcal{M}$
- $H = R \cap L$
- $D = R \circ L = L \circ R$
- $H \subseteq R, L \subseteq D \subseteq J$
Let $K$ be a field. Consider the monoid $\mathcal{M} = (M_n(K), \cdot)$ of square matrices of order $n$.

**Theorem**

For $A, B \in M_n(K)$, it holds that:

(i) $A \lessdot B \iff \row_K(A) = \row_K(B) \iff \ker A = \ker B$;
(ii) $A \rhd B \iff \col_K(A) = \col_K(B) \iff \im A = \im B$;
(iii) $A \join B \iff \rk A = \rk B$;
(iv) $A \heart B \iff \row_K(A) = \row_K(B)$ and $\col_K(A) = \col_K(B)$;
(v) $A \downarrow B \iff A \join B$, i.e., $\downarrow = \join$. 
Linear maps

**Definition**
A linear map $T : M_n(K) \to M_n(K)$ preserves a relation $\mathcal{X}$ (on $M_n(K)$) if $A \mathcal{X} B \Rightarrow T(A) \mathcal{X} T(B)$ for all $A, B \in M_n(K)$.

Which of the linear maps preserve Green’s relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D}$?
A linear map \( T: M_n(K) \to M_n(K) \) preserves a relation \( \mathcal{X} \) (on \( M_n(K) \)) if \( A \mathcal{X} B \Rightarrow T(A) \mathcal{X} T(B) \) for all \( A, B \in M_n(K) \).

Which of the linear maps preserve Green’s relations \( \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D} \)?

- \( T \equiv 0 \)
- \( T(A) = PAQ \), where \( P, Q \in GL_n(K) \). Preserves \( \mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D} \)
- \( T(A) = A^T \). Preserves \( \mathcal{H}, \mathcal{J}, \mathcal{D} \)
- \( T(A) = AX \), where \( X \in M_n(K) \). Preserves \( \mathcal{L} \)  
  \( (\text{Ker} \ A = \text{Ker} \ B \Rightarrow \text{Ker} \ AX = \text{Ker} \ BX) \)
- \( T(A) = XA \), where \( X \in M_n(K) \). Preserves \( \mathcal{R} \)
- Compositions of the above transformations
Let $S$ be a semifield: $\mathbb{R}_+, \mathbb{B}$ (boolean semiring), $\langle \mathbb{R} \cup \{-\infty\}, \max, + \rangle$ (tropical semifield), \ldots

A semifield is either anti-negative (any element except 0 does not have an additive inverse) or a field.
Let $S$ be a semifield: $\mathbb{R}_+, \mathbb{B}$ (boolean semiring), $\langle \mathbb{R} \cup \{ -\infty \}, \max, + \rangle$ (tropical semifield), ... A semifield is either anti-negative (any element except 0 does not have an additive inverse) or a field.

**Theorem (Guterman, Johnson, Kambites, 2018)**

Let $S$ be a semifield which is not a field and $T: M_n(S) \rightarrow M_n(S)$ a bijective $S$-linear map. The following are equivalent:

(i) $T$ preserves $\mathcal{L}$;

(ii) $T$ preserves $\mathcal{R}$;

(iii) There exist invertible (monomial) matrices $P, Q \in M_n(S)$ such that $T(A) = PAQ$ for all $A \in M_n(S)$.

**Theorem (Guterman, Johnson, Kambites, 2018)**

Let $S$ be a semifield which is not a field and $T: M_n(S) \rightarrow M_n(S)$ a bijective $S$-linear map. The following are equivalent:

(i) $T$ preserves $\mathcal{J}$;

(ii) $T$ preserves $\mathcal{D}$;

(iii) $T$ preserves $\mathcal{H}$;

(iv) There exist invertible (monomial) matrices $P, Q \in M_n(S)$ such that $T(A) = PAQ$ for all $A \in M_n(S)$ or $T(A) = PA^TQ$ for all $A \in M_n(S)$. 
Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $K$ be a field and $T: \mathcal{M}_n(K) \to \mathcal{M}_n(K)$ a linear map. The following are equivalent:

(i) $T$ preserves $\mathcal{J}$;

(ii) Either $T \equiv 0$, or there exist $P, Q \in \text{GL}_n(K)$ such that $T(A) = PAQ$ for all $A \in \mathcal{M}_n(K)$ or $T(A) = PATQ$ for all $A \in \mathcal{M}_n(K)$.

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(iii) There exist $P, Q \in GL_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$.

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $n \geq 1$ and $K$ be a field in which every polynomial of degree exactly $n$ has a root. The linear maps preserving $\mathcal{L}$ on $M_n(K)$ (resp. $\mathcal{R}$) are precisely those of the form $A \mapsto PAX$ (resp. $A \mapsto XAP$), where $P \in GL_n(K)$ and $X \in M_n(K)$.
Proof tools

**Theorem (Marcus and Moyls 1959; Minc 1977)**

Let $K$ be an algebraically closed field and $T: M_n(K) \to M_n(K)$ a linear map preserving the set of rank-1 matrices. Then $\exists P, Q \in GL_n(K)$:

- $T(A) = PAQ$ for all $A \in M_n(K)$ or
- $T(A) = PA^TQ$ for all $A \in M_n(K)$.

**Theorem (Lautemann 1981)**

Let $K$ be an arbitrary field and $T: M_n(K) \to M_n(K)$ a bijective linear map preserving the set of rank-1 matrices. Then $\exists P, Q \in GL_n(K)$:

- $T(A) = PAQ$ for all $A \in M_n(K)$ or
- $T(A) = PA^TQ$ for all $A \in M_n(K)$. 
### Statement

For $K = \mathbb{Q}$ and every $n \geq 2$, there exist non-bijective linear $\mathcal{L}$-preservers that do not fit the conditions of the above theorems.

### Example (Botta, 1978)

Let $f(x) \in K[x]$ be an irreducible polynomial of degree $n \geq 2$ (if exists). Let $C$ be any matrix in $M_n(K)$ whose minimal polynomial is $f(x)$. Then

$$\det(\lambda_1 I + \lambda_2 C + \lambda_3 C^2 + \ldots + \lambda_n C^{n-1}) = 0 \iff \lambda_1 = \lambda_2 = \ldots = \lambda_n = 0.$$

Hence the linear transformation $T$ defined as $T(A) = \sum_{i=1}^{n} a_{i1} C^{i-1}$, where $A = (a_{ij}) \in M_n(K)$, preserves $\mathcal{L}$ and $\mathcal{H}$, and is not of the form mentioned in the above theorems.
Examples

Statement

If $K = \mathbb{R}$ and $n$ is even, then there exist non-bijective linear $\mathcal{L}$-preservers that do not fit the conditions of the above theorems.

Example (based on a construction of Petrović, 2009)

Let $n$ be even, $n = 2m$.

$C_{2k-1} = E_{2k-1,1} + E_{2k,2}$ and $C_{2k} = E_{2k,1} - E_{2k-1,2}, \quad k = 1, \ldots, \frac{n}{2}$.

$T : M_n(K) \to M_n(K)$ is given by

\[
T(A) = \sum_{i=1}^{n} a_{i1} C_i = \begin{pmatrix}
a_{1,1} & -a_{2,1} & 0 & \cdots & 0 \\
a_{2,1} & a_{1,1} & 0 & \cdots & 0 \\
a_{3,1} & -a_{4,1} & 0 & \cdots & 0 \\
a_{4,1} & a_{3,1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{2m-1,1} & -a_{2m,1} & 0 & \cdots & 0 \\
a_{2m,1} & a_{2m-1,1} & 0 & \cdots & 0
\end{pmatrix}
\]
Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $K$ be a field and $T: M_n(K) \to M_n(K)$ a bijective linear map. The following are equivalent:

(i) $T$ preserves $\mathcal{H}$;

(ii) $T$ preserves $\mathcal{J}$;

(iii) There exist $P, Q \in \text{GL}_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PATQ$ for all $A \in M_n(K)$.

Theorem (Guterman, Johnson, Kambites, M., 2021)

Let $n \geq 1$ and $K$ be a field in which every polynomial of degree exactly $n$ has a root. Then for a linear map $T$ preserving the $\mathcal{H}$-relation on $M_n(K)$, it holds that either $T \equiv 0$, or there exist $P, Q \in \text{GL}_n(K)$ such that $T(A) = PAQ$ for all $A \in M_n(K)$ or $T(A) = PATQ$ for all $A \in M_n(K)$. 
Theorem (Guterman, Johnson, Kambites, M., 2021)

Let \( K \) be a field and \( T : M_n(K) \rightarrow M_n(K) \) a linear map. Then \( T \) preserves \( \mathcal{H} \) \( \iff \) \( T = 0 \) or \( T \) preserves the set of non-singular matrices.

Theorem (de Seguins Pazzis, 2010)

Let \( n \geq 2 \), \( K \) be any field, and \( T : M_n(K) \rightarrow M_n(K) \) be a linear non-singularity preserver. Then:

(i) either \( T \) is bijective and then there exist \( P, Q \in GL_n(K) \) such that \( T(A) = PAQ \) for all \( A \in M_n(K) \) or \( T(A) = PA^TQ \) for all \( A \in M_n(K) \);

(ii) or there exist an \( n \)-dimensional subspace \( V \subset M_n(K) \) contained in \( GL_n(K) \cup \{0_{n \times n}\} \), an isomorphism \( \alpha : K^n \rightarrow V \), and a non-zero \( x \in K^n \) such that

\[
T(M) = \alpha(Mx^T) \quad \forall M \in M_n(K) \quad \text{or} \quad T(M) = \alpha(M^Tx^T) \quad \forall M \in M_n(K)
\]
There exist non-bijective maps preserving $\mathcal{H}$ (and also $\mathcal{L}$).

**Example**

Let $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be defined as follows:

$$T \begin{pmatrix} a & \ast \\ b & \ast \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

This is the matrix representation of complex numbers.
Examples over $\mathbb{R}$

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Example

Let $T: M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be defined as follows:

$$T \begin{pmatrix} a & * \\ b & * \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

This is the matrix representation of complex numbers.

Example

Let $T: M_4(\mathbb{R}) \to M_4(\mathbb{R})$ be defined as follows:

$$T \begin{pmatrix} a & * & * & * \\ b & * & * & * \\ c & * & * & * \\ d & * & * & * \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} = aP_1 + bP_2 + cP_3 + dP_4,$$

where $P_1 = I_4$,

$$P_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

This is the matrix representation of quaternions.
Theorem (Kervaire, Milnor–Bott, 1958)

Any finite dimensional division algebra over the real numbers has dimension 1, 2, 4 or 8.

Corollary

If \( n \notin \{2, 4, 8\} \), then every linear \( \mathcal{H} \)-preserver on \( M_n(\mathbb{R}) \) is either zero or bijective.
Thanks for attention!